# PROBLEMS CLASS 8 5/3/2023

**Ex 26**

**Notation:** $Mat_n(\mathbb{F})$ set of $n \times n$ matrices with entries $\in \mathbb{F}$

$\mathbb{1}_n$ $n \times n$ identity matrix

$0_n$ " zero "

**Classical compact simple Lie groups:**

$$SU(N) = \{ g \in Mat_N(\mathbb{C}) \mid g^\dagger g = \mathbb{1}_N, \ \det g = 1 \}$$

$$SO(N) = \{ g \in Mat_N(\mathbb{R}) \mid g^T g = \mathbb{1}_N, \ \det g = 1 \}$$

$$USp(2N) = \{ g \in Mat_{2N}(\mathbb{C}) \mid g^\dagger g = \mathbb{1}_{2N}, \ g^T J g = J \}$$

where $J = \begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix} = -J^T$ "symplectic form".

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1. Characterize the Lie algebras $su(N)$, $so(N)$, $usp(2N)$ as vector spaces of matrices subject to linear conditions, which you should find.

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Write $g = e^{i\alpha} = \mathbb{1} + i\alpha + O(\alpha^2)$, where $\alpha \in$ Lie algebra of $G$.

- $su(N)$: $(\mathbb{1}_N - i\alpha^\dagger)(\mathbb{1}_N + i\alpha) + O(\alpha^2) = \mathbb{1}_N$

$$\mathbb{1}_N + i(\alpha - \alpha^\dagger) + O(\alpha^2) = \mathbb{1}_N$$

$$\Rightarrow \alpha = \alpha^\dagger \quad \leftarrow N \times N \text{ hermitian matrices}$$

$$1 = \det(e^{i\alpha}) \underset{\substack{\uparrow \\ \text{linear algebra}}}{=} e^{i\,tr(\alpha)} \quad \Rightarrow tr(\alpha) = 0 \quad \leftarrow \text{traceless}$$

$$\Rightarrow su(N) = \{ \alpha \in Mat_N(\mathbb{C}) \mid \alpha^\dagger = \alpha, \ tr(\alpha) = 0 \}.$$

• $so(N)$: $\mathbb{1}_N = (\mathbb{1}_N + i\alpha^T)(\mathbb{1}_N + i\alpha) + \mathcal{O}(\alpha^2)$
$= \mathbb{1}_N + i(\alpha + \alpha^T) + \mathcal{O}(\alpha^2)$
$\Rightarrow \alpha^T = -\alpha$ ← $N\times N$ antisymmetric matrices

$1 = \det(e^{i\alpha}) = e^{i\,tr(\alpha)} \Rightarrow tr(\alpha) = 0$, but this follows from $\alpha^T = -\alpha$.

$\Rightarrow so(N) = \{\alpha \in Mat_N(\mathbb{R}) \mid \alpha^T = -\alpha\}$

Note: $so(N,\mathbb{C}) = \{$ " $\mathbb{C}$ " $\}$

• $usp(2N)$: $\alpha^\dagger = \alpha$ ← $(2N)\times(2N)$ hermitian matrices

$J = (\mathbb{1}_{2N} + i\alpha^T) J (\mathbb{1}_{2N} + i\alpha) + \mathcal{O}(\alpha^2)$
$= J + i(\alpha^T J + J\alpha) + \mathcal{O}(\alpha^2)$
$\Rightarrow \alpha^T J = -J\alpha \Leftrightarrow (J\alpha)^T = J\alpha$

$\Rightarrow usp(2N) = \{\alpha \in Mat_{2N}(\mathbb{C}) \mid \alpha^\dagger = \alpha, (J\alpha)^T = J\alpha\}$

Note: $sp(2N) = \{\alpha \in Mat_{2N}(\mathbb{C}) \mid (J\alpha)^T = J\alpha\}$ (drop $g^\dagger g = \mathbb{1}_{2N}$ from definition of $Sp(2N$)

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2. Find the generators of the fundamental rep (fund) and its complex conjugate antifundamental rep ($\overline{\text{fund}}$) for $G = SU(N), SO(N), USp(2N)$.

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Exercise for the reader (tedious but instructive).
Hint: use the generators $e^{kl}$ of $gl(N)$, with $(e^{kl})_{ij} = \delta^k_i \delta^l_j$.

3. For $G = SO(N)$, $USp(2N)$, show that fund and $\overline{\text{fund}}$ are equivalent representations, namely

$$t_a^{(\overline{\text{fund}})} = V\, t_a^{(\text{fund})}\, V^{-1} \qquad \forall a$$

for some invertible matrix $V$ that you should find.

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Let's work directly at the level of groups, and show that

$$\underbrace{r^{(\overline{\text{fund}})}(g)}_{\equiv\ \bar{g}} = V\,\underbrace{r^{(\text{fund})}(g)}_{\equiv\ g}\,V^{-1}$$

(This implies the above $\forall$ generators)

- $G = SO(N)$:

($\phi \mapsto g\phi$ for $\phi \in$ fund)

$$\bar{g} = g \qquad \text{because } g \in \mathrm{Mat}_N(\mathbb{R}) \qquad \Rightarrow V = \mathbb{1}.$$

- $G = USp(2N)$:

$$\bar{g} = (g^{-1})^T = J g J^{-1} \qquad \Rightarrow V = J.$$

$$g^T J g = J \Leftrightarrow Jg = (g^{-1})^T J \Leftrightarrow Jg\underbrace{J^{-1}}_{\equiv\ -J} = (g^{-1})^T$$

Ex 35

Let the gauge group G be one of the classical compact simple Lie groups in Ex. 26. For each of the three infinite families $G = SU(N),\ SO(N),\ USp(2N)$:

NOT TODAY

1. Write down the finite (and infinitesimal) gauge transformations for the gauge field $A_\mu$, the field strength $F_{\mu\nu}$, a scalar field $\chi$ in fund rep, a scalar field $\Phi$ in adj rep.

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- $G = SU(N)$:

$$A_\mu \longmapsto A'_\mu = gA_\mu g^{-1} + ig(\partial_\mu g^{-1}) = gA_\mu g^{+} + ig(\partial_\mu g^{+})$$

$$F_{\mu\nu} \longmapsto F'_{\mu\nu} = gF_{\mu\nu}g^{-1} = gF_{\mu\nu}g^{+}$$

$$\chi \longmapsto \chi' = g\chi \qquad (\Leftrightarrow \chi^{+} \longmapsto \chi'^{+} = \chi g^{+} = \chi g^{-1})$$

$$\Phi \longmapsto \Phi' = g\Phi g^{-1} = g\Phi g^{+}$$

- $G = SO(N)$:

$$A_\mu \mapsto A'_\mu = gA_\mu g^{-1} + ig(\partial_\mu g^{-1}) = gA_\mu g^{T} + ig(\partial_\mu g^{T})$$
$$F_{\mu\nu} \mapsto F'_{\mu\nu} = gF_{\mu\nu}g^{T}$$
$$\chi \mapsto \chi' = g\chi \qquad (\Leftrightarrow \chi^{T} \mapsto \chi'^{T} = \chi g^{T} = \chi g^{-1})$$
$$\phi \mapsto \phi' = g\phi g^{T}$$

- $G = USp(2N)$: same, with $g^{-1} = g^{+} = -Jg^{T}J$.

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2. Write down a gauge invariant $\mathcal{L}$ for the above fields, including kinetic terms for the gauge field and the scalar fields, and mass terms for the scalar fields. You can ignore the theta term and assume the reality property

$$\chi = \bar{\chi} \qquad \text{for } G = SO(N)$$
$$\chi = J\bar{\chi} \qquad \text{for } G = USp(2N)$$

$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{scalars}$, where $\mathcal{L}_{YM} = -\frac{1}{2g_{YM}^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ in all cases.

• G=SU(N):

$$\mathcal{L}_{scalars} = -(D_\mu\chi)^\dagger(D^\mu\chi) - m_\chi^2\,\chi^\dagger\chi$$
$$-\,\mathrm{tr}((D_\mu\phi)^\dagger(D^\mu\phi)) - m_\phi^2\,\mathrm{tr}(\phi^\dagger\phi)$$

where $D_\mu\chi = (\partial_\mu - iA_\mu)\chi$, $(D_\mu\chi)^\dagger = \chi^\dagger(\overleftarrow{\partial}_\mu + iA_\mu) = \partial_\mu\chi^\dagger + i\chi^\dagger A_\mu$

$D_\mu\phi = \partial_\mu\phi - i[A_\mu,\phi]$, $(D_\mu\phi)^\dagger = \partial_\mu\phi^\dagger + i[\phi^\dagger, A_\mu] = \partial_\mu\phi^\dagger - i[A_\mu,\phi^\dagger]$

Finish the exercise for the other groups!