PROBLEMS CLASS 5

$\boxed{\text{Ex 2}}$ $\underline{G = U(1)}$ $D_\mu = \partial_\mu - iA_\mu$

$$[D_\mu, D_\nu] = -iF_{\mu\nu} = -i(\partial_\mu A_\nu - \partial_\nu A_\mu) \leftarrow$$

- $[\partial_\mu, \partial_\nu]f = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)f = \underset{\frac{\partial^2 f}{\partial x^\mu \partial x^\nu}}{\underset{\shortparallel}{(\partial_\mu\partial_\nu f)}} - \underset{\frac{\partial^2 f}{\partial x^\nu \partial x^\mu}}{\underset{\shortparallel}{(\partial_\nu\partial_\mu f)}} = 0$ $\forall f$: smooth test fn $\Leftrightarrow \underline{[\partial_\mu, \partial_\nu] = 0}$.

- $[g, h]f = ghf - hgf = 0$ $\forall f$ $\Leftrightarrow \underline{[g, h] = 0}$.

  ↑ ↑ functions; ↑ product of fns is commutative

- $[\partial_\mu, h]f = \partial_\mu(hf) - h(\partial_\mu f) = (\partial_\mu h)f + \cancel{h(\partial_\mu f)} - \cancel{h(\partial_\mu f)} = (\partial_\mu h)\cdot f$ $\forall f$

$$\Leftrightarrow \underline{[\partial_\mu, h] = (\partial_\mu h)} \qquad (\partial_\mu h) = \frac{\partial h}{\partial x^\mu}$$

$$[D_\mu, D_\nu] = [\partial_\mu - iA_\mu, \partial_\nu - iA_\nu] = \underset{0}{\underset{\shortparallel}{[\partial_\mu, \partial_\nu]}} - i\underbrace{[\partial_\mu, A_\nu]}_{=(\partial_\mu A_\nu)} - i\underbrace{[A_\mu, \partial_\nu]}_{=-(\partial_\nu A_\mu)} - \underset{0}{\underset{\shortparallel}{[A_\mu, A_\nu]}}$$

$$= -i(\partial_\mu A_\nu - \partial_\nu A_\mu).$$

Ex 14 Finite gauge transfo's:

$$\phi \mapsto \phi' = e^{i\alpha}\phi \qquad \phi \in \text{rep } \underline{r} \qquad \alpha^{(r)} = \alpha^a t_a^{(r)}$$
$$A_\mu \mapsto A'_\mu = e^{i\alpha} A_\mu e^{-i\alpha} + i e^{i\alpha}(\partial_\mu e^{-i\alpha})$$
$$F_{\mu\nu} \mapsto F'_{\mu\nu} = e^{i\alpha} F_{\mu\nu} e^{-i\alpha}$$

Taylor expand for small α:

$$e^{i\alpha} = \mathbb{1} + i\alpha + O(\alpha^2) \qquad X \mapsto X' \equiv X + \underline{\delta_\alpha X} + O(\alpha^2)$$

$$\phi' = \phi + \underline{i\alpha\phi} + O(\alpha^2) \qquad \Rightarrow \delta_\alpha \phi = i\alpha\phi$$

$$A'_\mu = (\mathbb{1} + i\alpha) A_\mu (\mathbb{1} - i\alpha) + i(\mathbb{1} + i\alpha)\underbrace{(\partial_\mu(\mathbb{1} - i\alpha))}_{-i\partial_\mu\alpha} + O(\alpha^2)$$

$$= A_\mu + \underline{i\alpha A_\mu - iA_\mu\alpha + i\cdot(-i)\partial_\mu\alpha} + O(\alpha^2)$$

$$\Rightarrow \delta_\alpha A_\mu = i[\alpha, A_\mu] + \partial_\mu\alpha .$$

$$F'_{\mu\nu} = (\mathbb{1} + i\alpha) F_{\mu\nu} (\mathbb{1} - i\alpha) + O(\alpha^2) = F_{\mu\nu} + \underline{i[\alpha, F_{\mu\nu}]} + O(\alpha^2)$$

$$\Rightarrow \delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$$

Ex 16

ϕ^a, $a = 1, \dots, \dim \mathfrak{g}$ $\underline{\text{adj}}$: $\phi \mapsto e^{i\alpha^a t_a^{(adj)}} \phi$

$$(t_a^{(adj)})^b{}_c = i f_{ac}{}^b$$

$A_\mu^{(adj)} = A_\mu^a t_a^{(adj)}$
$F_{\mu\nu}^{(adj)} = F_{\mu\nu}^a t_a^{(adj)}$

16.1

$$(A_\mu \phi)^a = (A_\mu^{(adj)})^a{}_c \phi^c = A_\mu^b (t_b^{(adj)})^a{}_c \phi^c$$

$$= A_\mu^b \cdot i f_{bc}{}^a \phi^c = i f_{bc}{}^a A_\mu^b \phi^c .$$

$$(F_{\mu\nu}\phi)^a = i f_{bc}{}^a F_{\mu\nu}^b \phi^c \qquad \text{(same logic)}$$

16.2

$$\Phi := \phi^a t_a, \quad A_\mu = A_\mu^a t_a, \quad F_{\mu\nu} = F_{\mu\nu}^a t_a .$$

- Show $(A_\mu\phi)^a t_a = [A_\mu, \Phi]$.

$$(A_\mu \phi)^a t_a = \underbrace{i f_{bc}{}^a t_a}_{= [t_b, t_c]} A_\mu^b \phi^c \qquad \text{Remember: } [t_a, t_b] = i f_{ab}{}^c t_c .$$

$$\overset{\text{bilinearity}}{=} [A_\mu^b t_b, \phi^c t_c] = [A_\mu, \Phi] . \qquad \square$$

- Show $D_\mu \Phi = \partial_\mu \Phi - i [A_\mu, \Phi]$

$$D_\mu \Phi \equiv (D_\mu \phi)^a t_a = (\partial_\mu \phi^a - i (A_\mu\phi)^a) t_a = \partial_\mu \underbrace{(\phi^a t_a)}_{= \Phi} - i \underbrace{(A_\mu \phi)^a t_a}_{= [A_\mu, \Phi]}$$

$$= \partial_\mu \Phi - i [A_\mu, \Phi] .$$

- Show $[D_\mu, D_\nu]\Phi = -i [F_{\mu\nu}, \Phi]$

Use $\underline{[D_\mu, D_\nu] = -i F_{\mu\nu}}$: $[D_\mu, D_\nu]\Phi = ([D_\mu, D_\nu]\Phi)^a t_a = -i (F_{\mu\nu}\Phi)^a t_a = -i [F_{\mu\nu}, \Phi]$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] .$$