PROBLEMS CLASS 6 – 8/2/2022

Ex 23

1. $$df(x) = \frac{\partial f(x)}{\partial x^\mu} dx^\mu = \frac{\partial f}{\partial x^\nu} dx^\nu$$

$$\tilde{x}^\mu = \tilde{x}^\mu(x) : \quad d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \quad \leftarrow dx^\mu \qquad 1)$$

$$\omega_\mu \mapsto \tilde{\omega}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \omega_\nu \qquad 2)$$

Show

$$\omega = \omega_\mu dx^\mu \longmapsto \tilde{\omega} = \tilde{\omega}_\mu d\tilde{x}^\mu = \omega$$

$$\tilde{\omega} = \tilde{\omega}_\mu d\tilde{x}^\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \omega_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\rho} dx^\rho = \underbrace{\frac{\partial x^\nu}{\partial x^\rho}}_{=\delta^\nu_\rho} \omega_\nu dx^\rho = \omega_\rho dx^\rho = \omega .$$

2. $$v = v^\mu \frac{\partial}{\partial x^\mu} = v^\mu \partial_\mu , \qquad \omega = \omega_\mu dx^\mu$$

$$\omega(v) = \omega_\mu dx^\mu (v^\nu \partial_\nu) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \omega_\mu v^\nu \underbrace{dx^\mu(\partial_\nu)}_{=\delta^\mu_\nu} = \omega_\mu v^\mu .$$

$\Vert$ $\omega = \tilde{\omega}$, $v = \tilde{v}$

$\tilde{\omega}(\tilde{v})$

$\cancel{\omega_\mu v^\mu = \frac{\partial x^\rho}{\partial x^\mu} \omega_\rho}$

$$\tilde{\omega}_\mu \tilde{v}^\mu = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \omega_\rho \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu = \frac{\partial x^\rho}{\partial x^\nu} \omega_\rho v^\nu$$
$$= \delta^\rho_\nu \omega_\rho v^\nu = \omega_\nu v^\nu$$

1. $g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$

$\forall\, u = u^\mu \partial_\mu ,\ v = v^\nu \partial_\nu$

$$g(u,v) = g(v,u)$$

$$(g_{\mu\nu}\, dx^\mu \otimes dx^\nu)(u,v) = (g_{\mu\nu}\, dx^\mu \otimes dx^\nu)(v,u)$$

$$g_{\mu\nu}\, dx^\mu(\underbrace{u}_{u^\rho \partial_\rho})\, dx^\nu(\underbrace{v}_{v^\sigma \partial_\sigma}) = g_{\mu\nu}\, dx^\mu(v)\, dx^\nu(u)$$

$$g_{\mu\nu}\, u^\rho \delta^\mu_\rho\, v^\sigma \delta^\nu_\sigma = g_{\mu\nu}\, v^\rho \delta^\mu_\rho\, u^\sigma \delta^\nu_\sigma$$

$\forall\, u^\rho, v^\sigma$ $$\underline{g_{\rho\sigma}\, u^\rho v^\sigma} = g_{\rho\sigma}\, v^\rho u^\sigma = \underline{g_{\sigma\rho}\, u^\rho v^\sigma}$$

$$\Longleftrightarrow \quad g_{\rho\sigma} = g_{\sigma\rho} .$$

$\boxed{\text{Ex } 25}$

1) $\mathbb{R}^2$: Euclidean (x, y) Polar (r, φ) $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$$g_{\mathbb{R}^2} = dx \otimes dx + dy \otimes dy$$

$$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi$$
$$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$$

$$g_{\mathbb{R}^2} = (\cos\varphi\, dr - r\sin\varphi\, d\varphi) \otimes (\cos\varphi\, dr - r\sin\varphi\, d\varphi)$$
$$+ (\sin\varphi\, dr + r\cos\varphi\, d\varphi) \otimes (\sin\varphi\, dr + r\cos\varphi\, d\varphi)$$
$$= dr \otimes dr\, [\cos^2\varphi + \sin^2\varphi] + d\varphi \otimes d\varphi\, [r^2\sin^2\varphi + r^2\cos^2\varphi]$$
$$+ dr \otimes d\varphi\, [-r\cos\cancel{\varphi\sin\varphi} + r\cancel{\cos\varphi}\sin\varphi] + d\varphi \otimes dr\, [\ 0\ ]$$

$$= dr \otimes dr + r^2 d\varphi \otimes d\varphi$$

$x^\mu = (x, y)$: $[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\tilde{x}^\mu \doteq (r, \varphi)$: $[\tilde{g}_{\mu\nu}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

cotangent vector (field) / "differential 1-form"

3. a

$$a = a_x\, dx + a_y\, dy = \underline{\underline{a_r}}\, dr + \underline{\underline{a_\varphi}}\, d\varphi$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi, \quad dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$$

$$a = a_x(\cos\varphi\, dr - r\sin\varphi\, d\varphi) + a_y(\sin\varphi\, dr + r\cos\varphi\, d\varphi)$$

$$= \underbrace{(a_x\cos\varphi + a_y\sin\varphi)}_{\equiv a_r} dr + \underbrace{(-r a_x\sin\varphi + r a_y\cos\varphi)}_{\equiv a_\varphi} d\varphi$$