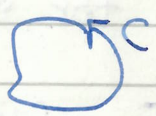
PROBLEMS CLASS 8 8/3/2022

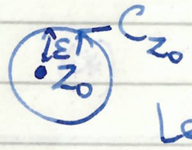
$\boxed{\text{Ex 32}}$ $\phi(\underline{x}) \in \mathbb{C}$, $\underline{x} \in \mathbb{R}^2 \cong \mathbb{C}$



$\mathbb{R}^2$

$$N[C] = \frac{1}{2\pi} \oint_C \nabla \arg(\phi) \cdot \underset{\substack{\shortparallel \\ d\underline{x}}}{d\underline{l}}$$

$z = x^1 + i x^2$



Let $\phi \approx \underset{\text{constants.}}{c\,(z - z_0)^n}$ near $z = z_0$. Show that $N[C_{z_0}] = n$.

Let $\underline{z = z_0 + \varepsilon e^{i\theta}}$, $\theta \in [0, 2\pi]$ to parametrize C_{z_0}. Then

$$\phi \approx c \cdot \varepsilon^n e^{in\theta}$$

$$\arg(\phi) = n\theta + \arg(c \cdot \varepsilon^n) = n\theta + \text{const}.$$

$$\nabla \arg(\phi) \cdot d\underline{l} = \partial_i \arg(\phi)\, dx^i = d\arg(\phi) = n\, d\theta$$

$$N[C_{z_0}] = \frac{n}{2\pi} \int_0^{2\pi} d\theta = n.$$

$\boxed{\text{Ex 34}}$ Global vortices

For static field configurations

$$E = \int d^2x \left[|\nabla\phi|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right].$$

$$\phi : \underset{\mathbb{C}}{\overset{\shortparallel}{\mathbb{R}^2}} \to \underset{\mathbb{C}}{\overset{\shortparallel}{\mathbb{R}^2}}$$

1) Let $\phi(\underline{x}) = \rho(\underline{x})\, e^{i\alpha(\underline{x})}$, $\rho, \alpha \in \mathbb{R}$. Show that

$$E = \int d^2x \left[(\nabla\rho)^2 + \rho^2(\nabla\alpha)^2 + \frac{\lambda}{2}(\rho^2 - v^2)^2 \right]. \qquad (*)$$

$$\nabla\phi = \nabla(\rho e^{i\alpha}) = (\nabla\rho)e^{i\alpha} + \rho e^{i\alpha}\cdot i(\nabla\alpha)$$

$$= e^{i\alpha}\left[\nabla\rho + i\rho\nabla\alpha\right]$$

$$|\nabla\phi|^2 = \underbrace{|e^{i\alpha}|^2}_{1} |\nabla\rho + i\rho\nabla\alpha|^2 = (\nabla\rho)^2 + \rho^2(\nabla\alpha)^2.$$

$$\frac{\lambda}{2}(|\phi|^2 - v^2)^2 = \frac{\lambda}{2}(\rho^2 - v^2)^2$$

$$\Rightarrow E = \int d^2x \left[(\nabla\rho)^2 + \rho^2(\nabla\alpha)^2 + \frac{\lambda}{2}(\rho^2 - v^2)^2 \right].$$

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2) Let $\boxed{\phi(\underline{x}) = f(r)\, e^{i\theta}}$ where $z = x^1 + ix^2 = re^{i\theta}$. Express E $(*)$ as an integral over r only, and show that

$$\rho^2(\nabla\alpha)^2 = \frac{f^2}{r^2}$$

and that, with the BC $f(\infty) = v$, this causes a logarithmic divergence of the integral that defines the static energy E.

We need an expression for $(\nabla\psi(r,\theta))^2$ in polar coordinates.

In cartesian coordinates $(x^1 = x,\ x^2 = y)$

$$(\nabla\psi)^2 = (\partial_x\psi)^2 + (\partial_y\psi)^2 . \quad \leftarrow \nabla\psi = (\partial_x\psi, \partial_y\psi).$$

$$\begin{cases} \partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta \\ \partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial\theta}{\partial y}\partial_\theta \end{cases} \quad \text{where} \quad \begin{matrix} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{matrix} \iff \begin{matrix} x = r\cos\theta \\ y = r\sin\theta . \end{matrix}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta \ , \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r} , \quad \frac{\partial\theta}{\partial y} = \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

$$\partial_x = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta \ , \quad \partial_y = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta$$

$$\Rightarrow (\nabla\psi)^2 = (\partial_x\psi)^2 + (\partial_y\psi)^2 = \left(\cos\theta\,\partial_r\psi - \frac{\sin\theta}{r}\partial_\theta\psi\right)^2 + \left(\sin\theta\,\partial_r\psi + \frac{\cos\theta}{r}\partial_\theta\psi\right)^2$$

$$= \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}(\partial_r\psi)^2 + \frac{1}{r^2}\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}(\partial_\theta\psi)^2 = (\partial_r\psi)^2 + \frac{1}{r^2}(\partial_\theta\psi)^2 .$$

- $\rho = f(r)$: $\quad (\nabla\rho)^2 = (\partial_r f(r))^2 + \frac{1}{r^2}\underbrace{(\partial_\theta f(r))^2}_{=0} = f'(r)^2$
- $\alpha = \theta$: $\quad (\nabla\alpha)^2 = \underbrace{(\partial_r\theta)^2}_{=0} + \frac{1}{r^2}(\partial_\theta\theta)^2 = \frac{1}{r^2}$

$$E = \int_0^\infty dr\, r \int_0^{2\pi} d\theta \left[f'(r)^2 + f(r)^2\frac{1}{r^2} + \frac{\lambda}{2}(f(r)^2 - v^2)^2 \right]$$

$$= 2\pi\int_0^\infty dr\, r \left[\qquad \text{"} \qquad \right]$$

$$E = 2\pi \int_0^\infty dr\, r \left[f'^2 + \frac{f^2}{r^2} + \frac{\lambda}{2}(f^2 - v^2)^2 \right] \qquad f = f(r)$$

BC as $r \to \infty$: $\underline{f \xrightarrow[r\to\infty]{} v = \text{const}}$

$$\int_\Lambda^R dr\, r \frac{f^2}{r^2} \underset{\substack{r>\Lambda\,:\, f(r)\approx v \\ \Lambda < R}}{\approx} \int_\Lambda^R dr\, r \frac{v^2}{r^2} = v^2 \int_\Lambda^R dr \frac{1}{r} = v^2 \log\frac{R}{\Lambda} \xrightarrow[R\to\infty]{} \infty$$

(Logarithmic divergence)

$\boxed{\text{Ex 35}}$ Static field(s) in D space dimensions $\phi(\underline{x})$

$$E[\phi] = \underbrace{\int d^Dx \, \frac{1}{2} G_{ab}(\phi)(\partial_i \phi^a)(\partial_i \phi^b)}_{= E_k[\phi]} + \underbrace{\int d^Dx \, V(\phi)}_{= E_v[\phi]}$$

Static energy ($\partial_0 \phi = 0$)

2) Let $\phi(\underline{x}) = \phi_1(\underline{x})$ be a static sol'n of the field eqns (EoM), and

$$\phi(\underline{x}) = \phi_\lambda(x) := \phi_1(\lambda x). \qquad \lambda > 0$$

Show that $E[\phi_\lambda] = \lambda^{2-D} E_k[\phi_1] + \lambda^{-D} E_v[\phi_1]$

- $E_v[\phi_\lambda] = \int d^Dx\, V(\phi_\lambda(x)) = \int d^Dx\, V(\phi_1(\lambda x)) \underset{y=\lambda x}{=} \lambda^{-D} \int d^Dy\, V(\phi_1(y))$

$$= \lambda^{-D} E_v[\phi_1] \qquad d^Dy = \lambda^D d^Dx$$

- $E_k[\phi_\lambda] = \int d^Dx \frac{1}{2} G_{ab}(\phi_\lambda(x)) \left(\frac{\partial}{\partial x^i} \phi^a_\lambda(x)\right)\left(\frac{\partial}{\partial x^i} \phi^b_\lambda(x)\right)$

$$= \int d^Dx \frac{1}{2} G_{ab}(\phi_1(\lambda x)) \left(\frac{\partial}{\partial x^i} \phi^a_1(\lambda x)\right)\left(\frac{\partial}{\partial x^i} \phi^b_1(\lambda x)\right)$$

$$\underset{y=\lambda x}{=} \lambda^{-D} \int d^Dy \frac{1}{2} G_{ab}(\phi_1(y)) \left(\lambda \frac{\partial}{\partial y^i} \phi^a_1(y)\right)\left(\lambda \frac{\partial}{\partial y^i} \phi^b_1(y)\right) = \lambda^{2-D} E_k[\phi_1]$$

$$\frac{\partial}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = \lambda \frac{\partial}{\partial y}$$