Q3 Let A_μ be a $U(1)$ gauge field, ϕ a scalar field of charge 1, and χ a scalar field of charge -2, with gauge transformations

$$(A_\mu,\ \phi,\ \chi) \ \mapsto\ (A_\mu + \partial_\mu \alpha,\ \exp[i\alpha]\phi,\ \exp[-2i\alpha]\chi)\ .$$

Show that

$$\chi(z) \exp\left[iq \int_{y,C}^{z} A_\mu(x) dx^\mu\right] \phi(y)^2$$

is gauge invariant for an appropriate value of the integer q that you should find. (The line integral in the exponent is over a curve C from point y to point z.)

Q4 It is given that, in a gauge where $A_0 = 0$, the energy of static field configurations of a gauge field A_μ and a scalar ϕ transforming in the adjoint representation of the gauge group is

$$E = \int d^3x \ \mathrm{tr}\left(\frac{1}{g_{YM}^2} B_i B_i + (D_i\phi)(D_i\phi)\right)\ ,$$

where the spatial indices $i = 1, 2, 3$ are summed over, D_i are covariant derivatives, $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, and the integral is over space $\mathbb{R}^3$. Find a lower bound for the energy E in terms of an integral over the 2-sphere at spatial infinity. You may use without proof the Bianchi identity $\epsilon_{ijk} D_i F_{jk} = 0$.

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$$(A_\mu, \phi, \chi) \longmapsto (A_\mu + \partial_\mu \alpha,\ e^{i\alpha}\phi,\ e^{-2i\alpha}\chi) \qquad (*)$$

- Let $x^\mu(\tau)$, $\tau \in [0,1]$, with $x^\mu(0) = y^\mu$, $x^\mu(1) = z^\mu$ be a parametrization of the curve C. Then

$$I_c(z,w) := \int_{y,C}^{z} A_\mu(x)\,dx^\mu = \int_0^1 d\tau\, \dot{x}^\mu(\tau)\, A_\mu(x(\tau))$$

transforms as follows under a gauge transformation $(*)$:

$$I_c(z,y) \longmapsto \int_0^1 d\tau\, \dot{x}^\mu(\tau) A_\mu(x(\tau)) + \int_0^1 d\tau\, \dot{x}^\mu(\tau)\, \partial_\mu \alpha(x(\tau))$$

$$= I_c(z,y) + \int_0^1 d\tau \frac{d}{d\tau}\alpha(x(\tau))$$

$$= I_c(z,y) + \Big[\alpha(x(\tau))\Big]_0^1$$

$$= I_c(z,y) + \alpha(x(1)) - \alpha(x(0))$$

$$= I_c(z,y) + \alpha(z) - \alpha(y).$$

- $\chi(z) \exp\left[iq \int_{y,C}^{z} A_\mu(x)\,dx^\mu\right] \phi(y)^2 = \chi(z)\, e^{iq I_c(z,y)}\, \phi(y)^2$

$$\underset{(*)}{\longmapsto} e^{-2i\alpha(z)} \chi(z)\, e^{iq\alpha(z) - iq\alpha(y)}\, e^{iq I_c(z,y)} \left[e^{i\alpha(y)}\phi(y)\right]^2$$

$$= \chi(z)\, e^{iq I_c(z,y)}\, \phi(y)^2$$

if $\underline{q=2}$.

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$$E = \int d^3x \; \mathrm{tr}\left(\frac{1}{g_{YM}^2} B_i B_i + (D_i\phi)(D_i\phi)\right)$$

ϕ in adjoint representation.

$$E \geq \int d^3x \; \mathrm{tr}\Big(\underbrace{\big(\tfrac{1}{g_{YM}}\vec{B} \mp \vec{D}\phi\big)^2}_{\geq 0} \pm \frac{2}{g_{YM}} \vec{B}\cdot\vec{D}\phi \Big)$$

$$\Rightarrow E \geq \pm \frac{2}{g_{YM}} \int d^3x \; \mathrm{tr}\,(\vec{B}\cdot\vec{D}\phi)$$

$$= \pm \frac{2}{g_{YM}} \int d^3x \; \mathrm{tr}\big(\vec{D}\cdot(\phi\vec{B})\big) \qquad (1)$$

($-\phi\vec{D}\cdot\vec{B} = 0$)

($\vec{D}\cdot\mathrm{tr}(\phi\vec{B})$)

$$= \pm \frac{2}{g_{YM}} \int d^3x \; \underbrace{\vec{\nabla}\cdot \mathrm{tr}(\phi\vec{B})}_{=\partial_i \mathrm{tr}(\phi B_i)} \qquad (2)$$

where we used

- the Bianchi identity $D_i B_i = \vec{D}\cdot\vec{B} = 0$ in (1)
- that $D_i\,\mathrm{tr}(\phi B_i) = \partial_i\,\mathrm{tr}(\phi B_i)$ in (2) since $\mathrm{tr}(\phi B_i)$ is gauge invariant.

$$\Rightarrow E \geq \frac{2}{g_{YM}} \left| \int_{\mathbb{R}^3} d^3x \; \vec{\nabla}\cdot \mathrm{tr}(\phi\vec{B}) \right|$$

$$= \frac{2}{g_{YM}} \left| \int_{S^2_\infty} dS \; \hat{n}\cdot \mathrm{tr}(\phi\vec{B}) \right|$$

(dS → $d^2\sigma$)

by Gauss' (/divergence) thm.

(We assumed $g_{YM} > 0$ wlog.)

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2. Calculate the magnetic flux through a 2-sphere of radius R centred at the origin,

$$\int_{S_R^2} \vec{B} \cdot d\vec{\sigma} = \int_{S_R^2} F_{\theta\varphi} d\theta d\varphi \ ,$$

where $\vec{B} = \vec{B}^{\pm} := \nabla \times \vec{A}^{\pm}$ in the regions where the two vector potentials $\vec{A}^{\pm}$ are defined, in both of the following two ways:

(a) By finding the magnetic field and integrating it over the 2-sphere;

(b) By reducing the surface integral over the 2-sphere to a line integral over its equator.

How does the flux depend on the radius R?

3. Show that the energy of this gauge field configuration

$$E = \frac{1}{2g^2} \int_{\mathbb{R}^3} d^3x \ \vec{B}^2 = \frac{1}{4g^2} \int_{\mathbb{R}^3} d^3x \ F_{ij} F^{ij}$$

is infinite. You may use without proof that $F^{\theta\varphi} = r^{-4} (\sin\theta)^{-2} F_{\theta\varphi}$.

Q5 (15 marks)
Scalar chromodynamics is a gauge theory with gauge group $G = SU(3)$ and N_f 'flavours' of scalar fields ϕ_i ($i = 1, \ldots, N_f$) transforming in the fundamental representation of the gauge group. NOTE: ϕ_i must be complex!

1. Write down a gauge invariant action including kinetic terms for the $SU(3)$ gauge field A_μ and the N_f scalars ϕ_i but no scalar potential, and check explicitly that this action is invariant under $SU(3)$ gauge transformations.
2. What is the global symmetry of the gauge theory with action written in part 1?
3. Write down the most general gauge invariant real scalar potential $V(\phi, \phi^\dagger)$ of degree at most 3 in ϕ and $\phi^\dagger$. (if $N_f=3$).
4. Repeat the exercise in part 3 under the further assumption that the global symmetry found in part 2 is preserved. (if $N_f=3$)

1. Gauge transfo's:

$$A_\mu \longmapsto A'_\mu = U A_\mu U^{-1} + i U(\partial_\mu U^{-1})$$

$$U \in SU(3)$$

$$\phi_i \longmapsto \phi'_i = U\phi_i$$

Gauge invariant kinetic action:

$$S_{kin} = \int d^4x \left[-\frac{1}{2g_{YM}^2} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{16\pi^2} \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) - \sum_{i=1}^{N_f} (D_\mu \phi_i)^\dagger (D^\mu \phi_i) \right]$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \longmapsto F'_{\mu\nu} = U F_{\mu\nu} U^{-1}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \longrightarrow \tilde{F}'^{\mu\nu} = U \tilde{F}^{\mu\nu} U^{-1}$$

$$D_\mu \phi_i = \partial_\mu \phi_i - i A_\mu \phi_i \longmapsto (D_\mu \phi_i)' = U D_\mu \phi_i$$

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \mapsto \mathrm{tr}(U F_{\mu\nu} U^{-1} U \overset{(\sim)}{F}{}^{\mu\nu} U^{-1}) = \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

by $U^{-1}U = \mathbb{1}$ and cyclicity of tr.

$$(D_\mu\phi_i)^\dagger (D^\mu \phi_i) \mapsto (U D_\mu \phi_i)^\dagger (U D^\mu \phi_i) = (D_\mu \phi_i)^\dagger \underset{U^{-1}}{\underset{\parallel}{U^\dagger}} U (D^\mu \phi_i) = (D_\mu\phi_i)^\dagger (D^\mu \phi_i)$$

2. Internal Global symmetry can act on flavour indices $\underset{j}{i} = 1, \dots, N_f$.

Including gauge and flavour symmetry transfo (V):

Of course the Spacetime symmetry is the POINCARÉ GROUP ISO(1,3)

$$\phi_i \mapsto \phi'_i = U \phi_i V^i{}_j \quad \text{or} \quad \Phi \equiv (\phi_1, \phi_2, \dots, \phi_{N_f}) \longrightarrow U\phi V$$

(U: x-dependent; V: constant)

$$= \begin{pmatrix} \phi_1^1 & \cdots & \phi_{N_f}^1 \\ \phi_1^2 & \cdots & \vdots \\ \phi_1^3 & \cdots & \phi_{N_f}^3 \end{pmatrix}$$

$$\sum_i (D_\mu\phi_i)^\dagger (D^\mu\phi_i) = \mathrm{tr}\left((D_\mu \Phi)^\dagger (D^\mu \Phi)\right)$$

↑ trace over flavour indices i,j

$$\downarrow$$

$$\mathrm{tr}\left(V^\dagger (D_\mu\Phi)^\dagger (D^\mu \Phi) V\right) = \mathrm{tr}\left(V V^\dagger (D_\mu\Phi)^\dagger (D^\mu\Phi)\right) = \mathrm{tr}\left((D_\mu\Phi)^\dagger (D^\mu\Phi)\right)$$

iff $VV^\dagger = \mathbb{1}_{N_f} \Rightarrow V \in U(N_f)$ ← "Internal global symmetry"

3. $V(\phi^\dagger, \phi) =:$ $\boxed{N_f = 3}$

degree 0 : $c \cdot 1$ ← constant $\in \mathbb{R}$

degree 1 : $a^{\dagger i} \phi_i + c.c.$ ← not gauge invariant ⇒ $a^i = 0$.

degree 2 : $M^{\dagger i}{}_j \bar{\Phi}^j \phi_i$, $M = M^\dagger$ constant matrix

degree 3: $\bar{\Phi}\phi\phi$, $\bar{\Phi}\bar{\Phi}\phi$ ← not gauge invariant

$$\underbrace{\lambda \, \varepsilon_{abc} \, \varepsilon^{ijk} \phi^a_i \phi^b_j \phi^c_k}_{\det(\Phi)} + c.c.$$

$$\det(\Phi) \mapsto \det(U \Phi V) = (\det U)(\det \Phi)(\det V) = (\det \Phi) \cdot (\det V)$$

$\mathrm{tr}(M\phi^\dagger\phi)$ ← (trace over flavour indices)

$$\Rightarrow V(\phi^\dagger, \phi) = c + M^i{}_j \bar{\Phi}^j \phi_i + (\lambda \det(\Phi) + c.c.)$$

with $c \in \mathbb{R}$, $M = M^\dagger$, $\lambda \in \mathbb{C}$.

4. • $\mathrm{tr}(M \Phi^\dagger \Phi) \mapsto \mathrm{tr}(M V^\dagger \Phi^\dagger U^\dagger U \Phi V) = \mathrm{tr}(V M V^\dagger \Phi^\dagger \Phi)$

only invariant under $U(3)$ flavour symmetry if $M = m \mathbb{1}_{3\times3}$.

• $\det \Phi \mapsto (\det \Phi) \cdot (\det V)$

not invariant under $U(3)$ flavour symmetry. (Invariant under $SU(3) \subset U(3)$ though)

$$\Rightarrow V(\phi^\dagger, \phi) = c + m \, \mathrm{tr}(\Phi^\dagger \Phi) \quad , \quad c, m \in \mathbb{R}.$$