Assignment 5

Due date: Monday, 30 January (9am)

Ex 9

Write down the most general real gauge invariant Lagrangian with at most two derivatives for two complex scalar fields, ϕ of charge 1 and χ of charge 2, and a $U(1)$ gauge field A_μ, which comprises:

1. kinetic terms for ϕ and χ; [15 marks]

   SOLUTION:

   Since the scalar fields are charged under a $U(1)$ gauge symmetry, we should replace the partial derivatives in the standard kinetic term $-\partial^\mu\bar{\phi}\partial_\mu\phi - \partial^\mu\bar{\chi}\partial_\mu\chi$ by gauge covariant derivatives. We need to remember that the gauge covariant derivative of a field of charge q is $D_\mu = \partial_\mu - iqA_\mu$. So we have the gauge invariant kinetic terms

   $$\mathcal{L}_{\text{kin}} = -(\partial^\mu\bar{\phi} + iA^\mu\bar{\phi})(\partial_\mu\phi - iA_\mu\phi) - (\partial^\mu\bar{\chi} + 2iA^\mu\bar{\chi})(\partial_\mu\chi - 2iA_\mu\chi)\ .$$

2. a kinetic term for A_μ; [10 marks]

   SOLUTION:

   This is simply the Maxwell Lagrangian density that we saw in the gauge theory formulation of electromagnetism:

   $$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu}$$

   where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

3. a real gauge invariant potential which is a polynomial of degree at most 4 in ϕ, χ and their complex conjugates. [25 marks]

   SOLUTION:

   We start by noticing that for any fields f_1, f_2 of charges q_1, q_2 respectively under a $U(1)$ symmetry (global or local/gauge), their product $f_1 f_2$ has charge $q_1 + q_2$. Indeed under a $U(1)$ transformation with group element $e^{i\alpha}$ the fields transform as

   $$(f_1, f_2) \mapsto (e^{iq_1\alpha}f_1, e^{iq_2\alpha}f_2) \qquad \Longrightarrow \qquad f_1 f_2 \mapsto e^{i(q_1+q_2)\alpha}f_1 f_2\ .$$

This generalizes by induction to monomials in the fields: the charge of a monomial is the sum of the charges of its factors. In particular, a monomial in the fields is invariant under a $U(1)$ gauge transformation if and only if it has charge 0.

The scalar potential is a polynomial in $\phi, \chi, \bar{\phi}, \bar{\chi}$. Demanding gauge invariance (*i.e.* vanishing total charge), we see that the only allowed monomials are

$$1 \ , \quad |\phi|^2 = \bar{\phi}\phi \ , \quad |\chi|^2 = \bar{\chi}\chi \ , \quad \bar{\chi}\phi^2 \ , \quad \bar{\phi}^2\chi$$

and products/powers thereof. Therefore the most general real gauge invariant potential which is a polynomial of degree at most 4 in ϕ, χ and their complex conjugates is

$$V(\bar{\phi}, \bar{\chi}, \phi, \chi) = V_0 + m_\phi^2|\phi|^2 + m_\chi^2|\chi|^2 + \mathrm{Re}(a\bar{\chi}\phi^2) + \lambda_\phi|\phi|^4 + \lambda_\chi|\chi|^4 + \lambda_{\phi\chi}|\phi|^2|\chi|^2 \ ,$$

where $V_0, m_\phi^2, m_\chi^2, \lambda_\phi, \lambda_\chi, \lambda_{\phi\chi}$ are real constants, and a is a complex constant. The constant V_0 (the 'vacuum energy density') is often ignored since it drops out of the equations of motion, and the energy is defined up to an additive constant.

Ex 10

Consider "scalar electrodynamics", the field theory with Lagrangian density

$$\mathcal{L} = -\overline{D_\mu\phi}D^\mu\phi - U(|\phi|^2) - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \ , \tag{1.1}$$

where

$$D_\mu\phi = (\partial_\mu - iA_\mu)\phi \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

1. Show that the equations of motion (Euler-Lagrange equations) for the complex scalar field ϕ and for the real $U(1)$ gauge field A_μ are

$$D_\mu D^\mu\phi = U'(|\phi|^2)\phi \ , \qquad \partial_\nu F^{\mu\nu} = g^2 J^\mu \ ,$$

where

$$J_\mu = -i(\bar{\phi}D_\mu\phi - \phi\overline{D_\mu\phi}) \ .$$

[30 marks]

SOLUTION:

We need to use the Euler-Lagrange equation, which for a field X read

$$0 = \frac{\partial\mathcal{L}}{\partial X} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \equiv \frac{\partial\mathcal{L}}{\partial X} - \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 X)} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i X)} \ ,$$

applied to $X = \bar{\phi}$ and $X = A_\nu$. (The Euler-Lagrange equation for $X = \phi$ is the complex conjugate of the Euler-Lagrange equation for $X = \bar{\phi}$, since the Lagrangian density is real.)

Let us first work out the equation of motion for ϕ, which is obtained by using $X = \bar{\phi}$ in the Euler-Lagrange equation above. The simplest way to proceed is perhaps to integrate by parts the kinetic term of ϕ in the action, or write the Lagrangian density as

$$\mathcal{L} = \bar{\phi} D_\mu D^\mu \phi - U(|\phi|^2) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \partial_\mu(\dots) \equiv \mathcal{L}' + \partial_\mu(\dots) \ .$$

The last term is a total derivative, which integrates to a boundary (or 'surface') term in the action, which in turn does not contribute to the equations of motion (which are obtained by setting to zero the first variation of the action under any variations of the fields, see Math Phys II or the first term). Then the E-L eqn becomes $\partial\mathcal{L}'/\partial\bar{\phi} = 0$, which leads to

$$D_\mu D^\mu \phi - U'(|\phi|^2)\phi = 0 \ .$$

Alternatively, let us write down the covariant derivative explicitly:

$$\mathcal{L} = -(\partial_\mu \bar{\phi} + i A_\mu \bar{\phi})(\partial^\mu \phi - i A^\mu \phi) - U(|\phi|^2) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \ .$$

Then

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\phi}} &= -i A_\mu (\partial^\mu \phi - i A^\mu \phi) - U'(|\phi|^2)\phi \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} &= -(\partial^\mu \phi - i A^\mu \phi) \ , \end{aligned}$$

which leads to the E-L equation

$$0 = (\partial_\mu - i A_\mu)(\partial^\mu - i A^\mu)\phi - U'(|\phi|^2)\phi \equiv D_\mu D^\mu \phi - U'(|\phi|^2)\phi \ .$$

The equation of motion for the gauge field ($X = A_\nu$) is a little more involved to derive, but we can make progress if we notice that A_ν only appears inside the covariant derivatives $D_\nu \phi$ and $D_\nu \bar{\phi}$, whereas $\partial_\mu A_\nu$ only appears inside the Maxwell term, which depends on the field strength. Using the chain rule and $\partial A_\mu / \partial A_\nu = \delta_\mu^\nu$, we calculate

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= \frac{\partial \mathcal{L}}{\partial(D_\mu \phi)} \frac{\partial(D_\mu \phi)}{\partial A_\nu} + \frac{\partial \mathcal{L}}{\partial(\overline{D_\mu \phi})} \frac{\partial(\overline{D_\mu \phi})}{\partial A_\nu} = -\overline{D^\mu \phi}(-i\delta_\mu^\nu \phi) + (c.c) \\ &= i(\phi \overline{D^\nu \phi} - \bar{\phi} D^\nu \phi) \equiv J^\nu \ . \end{aligned}$$

Then we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} \ ,$$

where we first view $F_{\rho\sigma}$ as independent variables that the Lagrangian density depends on, and then express them as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Let's compute the two factors separately. Being explicit with indices,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} &= -\frac{1}{4g^2} \eta^{\mu\alpha} \eta^{\nu\beta} \frac{\partial}{\partial F_{\rho\sigma}} (F_{\mu\nu} F_{\alpha\beta}) = -\frac{1}{4g^2} \eta^{\mu\alpha} \eta^{\nu\beta} \left(\frac{\partial F_{\mu\nu}}{\partial F_{\rho\sigma}} F_{\alpha\beta} + F_{\mu\nu} \frac{\partial F_{\alpha\beta}}{\partial F_{\rho\sigma}} \right) \\ &= -\frac{1}{4g^2} \eta^{\mu\alpha} \eta^{\nu\beta} \left(\delta_\mu^\rho \delta_\nu^\sigma F_{\alpha\beta} + F_{\mu\nu} \delta_\alpha^\rho \delta_\beta^\sigma \right) = -\frac{1}{2g^2} F^{\rho\sigma} \ . \end{aligned}$$

(Having done this exercise once, later on we will use $\frac{\partial}{\partial X_\nu}(X^\mu X_\mu) = 2X^\nu$, $\frac{\partial}{\partial X_{\rho\sigma}}(X^{\mu\nu} X_{\mu\nu}) = 2X^{\rho\sigma}$ *etc.* without further proof.) For the second factor,

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \ .$$

Putting the previous results together, we find

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2g^2} F^{\rho\sigma} \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho\right) = -\frac{1}{g^2} F^{\mu\nu} = \frac{1}{g^2} F^{\nu\mu}$$

So the equation of motion for the gauge field reads

$$0 = J^\nu - \partial_\mu \left(\frac{1}{g^2} F^{\nu\mu}\right) \quad \Longleftrightarrow \quad \partial_\mu F^{\nu\mu} = g^2 J^\nu \ .$$

2. Show that the current J_μ is real, gauge invariant, and conserved ($\partial_\mu J^\mu = 0$) upon using the equations of motion. [15 marks]

SOLUTION:

Start by writing $J_\mu = 2\text{Im}(\bar{\phi} D_\mu \phi)$, which is manifestly real (Im denotes the imaginary part of a complex number). Under a $U(1)$ gauge transformation $\phi \mapsto e^{i\alpha}\phi$, which implies $\bar{\phi} \mapsto e^{-i\alpha}\bar{\phi}$ by complex conjugation, and $D_\mu \phi \mapsto e^{i\alpha} D_\mu \phi$ since D_μ is a covariant derivative (quoting this result is fine, rederiving it is even better). So

$$J_\mu = 2\ \text{Im}(\bar{\phi} D_\mu \phi) \mapsto 2\ \text{Im}(\bar{\phi} e^{-i\alpha} e^{i\alpha} D_\mu \phi) = 2\ \text{Im}(\bar{\phi} D_\mu \phi) = J_\mu \ .$$

To see that J_μ is conserved, use the equation of motion for the gauge field and the antisymmetry of the field strength and the commutativity of partial derivatives:

$$\partial_\nu J^\nu = \frac{1}{g^2} \partial_\nu \partial_\mu F^{\nu\mu} = -\frac{1}{g^2} \partial_\nu \partial_\mu F^{\mu\nu} = -\frac{1}{g^2} \partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\nu J^\nu \ ,$$

which implies $\partial_\nu J^\nu = 0$.

Alternatively, use the explicit form of J^μ and calculate its divergence. (It might help to use $D_\mu J^\mu = \partial_\mu J^\mu$, which follows from the fact that J^μ is gauge invariant (in other words, it has charge zero), and $\overline{D_\mu \phi} = D_\mu \bar{\phi}$, where in the last expression the covariant derivative is understood to be in the representation of charge -1, since $\bar{\phi}$ has opposite charge to ϕ. Then

$$\partial_\mu J^\mu = D_\mu J^\mu = -i \left((D_\mu \bar{\phi})(D_\mu \phi) + \bar{\phi}(D_\mu D^\mu \phi) - (D_\mu \phi)(D^\mu \bar{\phi}) - \phi(D_\mu D^\mu \bar{\phi}) \right)$$

The first and the third term are equal and opposite and cancel out; the second and the fourth term cancel out upon using the equation of motion for ϕ and the complex conjugate equation of motion for $\bar{\phi}$ (noting that the scalar potential U is real).

3. Show that the conserved current J_μ in the gauge theory and the conserved current j_μ in the scalar field theory with $U(1)$ global symmetry, which is obtained by setting $A_\mu = 0$ in the Lagrangian (1.1), are related by

$$J_\mu = j_\mu + bA_\mu|\phi|^2 ,$$

for a constant b that you should find. [5 marks]

SOLUTION:

$$\begin{aligned} J_\mu &= 2\ \text{Im}(\bar{\phi}D_\mu\phi) = 2\ \text{Im}\left(\bar{\phi}(\partial_\mu - iA_\mu)\phi\right) = 2\ \text{Im}(\bar{\phi}\partial_\mu\phi) + 2\ \text{Im}(-iA_\mu|\phi|^2) \\ &= 2\ \text{Im}(\bar{\phi}\partial_\mu\phi) + 2A_\mu|\phi|^2\text{Im}(-i) = j_\mu - 2A_\mu|\phi^2| , \end{aligned}$$

where we used that A_μ and $|\phi|^2 = \bar{\phi}\phi$ are real. Hence $b = -2$.