Assignment 6

Due date: Monday, 13 February (noon)

Ex 20

Let $\phi^{1,2}(x)$ be two real scalar fields in two space and one time dimensions (x^0, x^1, x^2), and $\phi(x) = \phi^1(x) + i\phi^2(x)$.

1. Show that the current
$$j^\mu = c\frac{1}{2}\epsilon_{ab}\epsilon^{\mu\nu\rho}\partial_\nu(\phi^a\partial_\rho\phi^b)$$
is conserved, that is $\partial_\mu j^\mu = 0$, regardless of the equations of motion. Here c is a normalization constant, $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor in three indices with $\epsilon^{012} = 1$, and ϵ_{ab} is the totally antisymmetric tensor in two indices with $\epsilon_{12} = 1$. [20 marks]

   SOLUTION:

   This uses material from last term (or last year) and is hopefully easy: we just calculate
$$\partial_\mu j^\mu = c\frac{1}{2}\epsilon_{ab}\epsilon^{\mu\nu\rho}\partial_\mu\partial_\nu(\phi^a\partial_\rho\phi^b) = 0 ,$$
which vanishes because $\epsilon^{\mu\nu\rho}$ is totally antisymmetric in its indices, and $\partial_\mu\partial_\nu(\phi^a\partial_\rho\phi^b)$ is symmetric in μ, ν if we assume, as we always do, that the scalar fields ϕ^a are sufficiently smooth so that partial derivatives commute.

2. Assume that $|\phi| \to v$ at spatial infinity, where v is a constant. Write down the conserved charge Q associated to the current j^μ, and show that Q is equal to the total winding number of the argument of ϕ,
$$N = \frac{1}{2\pi}\int_{S^1_\infty} \nabla \arg(\phi) \cdot d\vec{l} ,$$
for a suitable choice of the normalization constant c that you should find. [30 marks]

SOLUTION:

The conserved charge Q is the integral over space (here, $\mathbb{R}^2$) of the charge density

$$\rho = j^0 = \frac{c}{2}\epsilon_{ab}\epsilon^{0\nu\rho}\partial_\nu(\phi^a\partial_\rho\phi^b) = \frac{c}{2}\epsilon_{ab}\epsilon^{ij}\partial_i(\phi^a\partial_j\phi^b) \ .$$

In the last equality we have used that $\epsilon^{0\nu\rho}$ does not vanish only if both ν and ρ are spatial indices, which we can call $i, j \in \{1, 2\}$, and $\epsilon^{0ij} = \epsilon^{ij}$, where $\epsilon^{12} = 1$. This charge density is the curl of a 2-vector $\vec{V} = (V_1, V_2)$ (recall that in two dimensions the curl of a vector is a scalar):

$$\rho = \nabla \times \vec{V} \equiv \epsilon^{ij}\partial_i V_j = \partial_1 V_2 - \partial_2 V_1 \ , \qquad V_j = \frac{c}{2}\epsilon_{ab}\phi^a\partial_j\phi^b = \frac{c}{2}(\phi^1\partial_j\phi^2 - \phi^2\partial_j\phi^1) \ .$$

Then by Stokes' theorem

$$Q = \int_{\mathbb{R}^2} d^2x \ \rho = \int_{\mathbb{R}^2} d^2x \ \nabla \times \vec{V} = \oint_{S^1_\infty} \vec{V} \cdot d\vec{l} \ .$$

There are a number of ways to proceed now. One is to use the complex field $\phi = \phi^1 + i\phi^2 = |\phi|e^{i\arg(\phi)}$, in terms of which

$$\phi^1\partial_j\phi^2 - \phi^2\partial_j\phi^1 = \mathrm{Im}(\bar{\phi}\partial_j\phi) = |\phi^2| \ \partial_j \arg(\phi) \ .$$

At spatial infinity $|\phi| \to v$, so

$$Q = \oint_{S^1_\infty} \vec{V} \cdot d\vec{l} = \frac{1}{2}cv^2 \oint_{S^1_\infty} \nabla \arg(\phi) \cdot d\vec{l} \ ,$$

therefore $Q = N$ if

$$cv^2 = \frac{1}{\pi} \quad \Longrightarrow \quad c = \frac{1}{\pi v^2} \ .$$

3. Let $z = x^1 + ix^2$ be a complex coordinate on the spatial plane. For each choice of the sign $\epsilon = \pm 1$, rewrite the Bogomol'nyi equations for the abelian Higgs model

$$(D_1 - i\epsilon D_2)\phi = 0$$
$$F_{12} = \epsilon g^2(|\phi|^2 - v^2)$$

in terms of the complex coordinates $(z, \bar{z})$ rather than (x^1, x^2). Solve the first Bogomol'nyi equation to determine the holomorphic and antiholomorphic components $A_z, A_{\bar{z}}$ of the gauge field (remember that A_μ is real if $\mu = x^1, x^2$). Substitute the result in the second Bogomol'nyi equation to obtain a partial differential equation for $|\phi|^2$ only.
[50 marks]

SOLUTION:

This final part is challenging, but I wanted to see how you perform in a more advanced and less guided question.[1] I'll give partial credit for partial work where it is due.

[1]This final part could become a section B exam question, with some extra help provided. The first two parts together could be a section A question, or perhaps an easier section B exam question.

From the change of variables between Euclidean and complex coordinates

$$x^1 = \frac{1}{2}(z+\bar{z}) \ , \qquad x^1 = -\frac{i}{2}(z-\bar{z})$$

and the chain rule we find

$$\partial_z \equiv \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right) \equiv \frac{1}{2}(\partial_1 - i\partial_2)$$
$$\partial_{\bar{z}} \equiv \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right) \equiv \frac{1}{2}(\partial_1 + i\partial_2) \ .$$

Since the gauge field A_μ and the partial derivative ∂_μ are linearly combined in the gauge covariant derivative $D_\mu = \partial_\mu - iA_\mu$, they should transform in the same way under a change of coordinates.[2] (Quoting what I stated in the lecture is also fine.) Therefore

$$A_z = \frac{1}{2}(A_1 - iA_2) \ , \qquad A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2) \ ,$$

and

$$D_z = \frac{1}{2}(D_1 - iD_2) \ , \qquad D_{\bar{z}} = \frac{1}{2}(D_1 + iD_2) \ .$$

Similarly, we calculate

$$F_{12} = \frac{\partial z}{\partial x^1}\frac{\partial \bar{z}}{\partial x^2}F_{z\bar{z}} + \frac{\partial \bar{z}}{\partial x^1}\frac{\partial z}{\partial x^2}F_{\bar{z}z} = \left(\frac{\partial z}{\partial x^1}\frac{\partial \bar{z}}{\partial x^2} - \frac{\partial \bar{z}}{\partial x^1}\frac{\partial z}{\partial x^2}\right)F_{z\bar{z}} = -2iF_{z\bar{z}}$$

where in the first equality we used the transformation properties of tensors under changes of coordinates, which I stated in the lecture, and in the second equality we used the antisymmetry of the field strength. (Using $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$ and the above expressions for $\partial_z, \partial_{\bar{z}}, A_z, A_{\bar{z}}$ to obtain $F_{z\bar{z}} = \frac{i}{2}F_{12}$ is also fine.)

If $\epsilon = +1$, the first Bogomol'nyi equation is $0 = D_z\phi = \partial_z\phi - iA_z\phi$, which determines

$$A_z = -i\frac{\partial_z\phi}{\phi} = -i\partial_z \log\phi \ .$$

Complex conjugating and using that A_1, A_2 are real, we find

$$A_{\bar{z}} = \overline{A_z} = i\frac{\partial_{\bar{z}}\bar{\phi}}{\bar{\phi}} = i\partial_{\bar{z}}\log\bar{\phi} \ .$$

Setting $X = |\phi|^2$ and $\sigma = \arg(\phi)$, we calculate

$$A_z = -\frac{i}{2}\partial_z \log X + \partial_z\sigma$$
$$A_{\bar{z}} = +\frac{i}{2}\partial_{\bar{z}} \log X + \partial_{\bar{z}}\sigma$$

[2] More precisely, this determines $A_z, A_{\bar{z}}$ up to a gauge transformation.

which implies (the last equality is not really necessary)

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = i\partial_z \partial_{\bar{z}} \log X = i\left(\frac{\partial_z \partial_{\bar{z}} X}{X} - \frac{|\partial_z X|^2}{X^2}\right) \ .$$

Putting everything together, the second Bogomol'nyi equation becomes

$$2\partial_z \partial_{\bar{z}} \log X = g^2(X - v^2)$$

where $X = |\phi|^2$. (To be very precise, there are also some delta functions supported at the points where $\arg(\phi)$ is ill-defined, namely where $\phi = 0, \infty$, but let's not worry about that.)

If $\epsilon = -1$, the roles of z and $\bar{z}$ are swapped. The field strength has the opposite sign and the second Bogomol'nyi equation reads

$$2\partial_z \partial_{\bar{z}} \log X = -g^2(X - v^2)$$

(again, except for some delta functions that are beyond the scope of this exercise).