Assignment 7

Due date: Friday, 3 March (8pm)

Ex 21

A magnetic monopole of magnetic charge m located at the origin O of three-dimensional space is described by a divergence-free magnetic field $\vec{B}$ in $\mathbb{R}^3 \setminus O$, with non-vanishing magnetic flux though the 2-sphere that surrounds the origin O:

$$\frac{1}{2\pi} \int_{S^2} \vec{B} \cdot d\vec{\sigma} = m \neq 0.$$

1. Show that all of the above can be reformulated as the equation

   $$\nabla \cdot \vec{B} = 2\pi m \; \delta^{(3)}(\vec{x})$$

   in $\mathbb{R}^3$. [10 marks]

   SOLUTION:

   If $\vec{x} \neq \vec{0}$ then $\nabla \cdot \vec{B} = 0$ is equivalent to $\nabla \cdot \vec{B} = 2\pi m \; \delta^{(3)}(\vec{x})$.

   If we extend $\mathbb{R}^3 \setminus O$ to $\mathbb{R}^3$, a 2-sphere S^2 (of any radius) surrounding the origin is the boundary of a 3-ball B^3 of the same radius, and Gauss' theorem gives

   $$\int_{S^2} \vec{B} \cdot d\vec{\sigma} = \int_{B^3} \nabla \cdot \vec{B} \; d^3x \; .$$

   The volume integral on the right-hand side can only receive contribution from the origin, because $\vec{B}$ is divergence-free elsewhere. (Alternatively, the flux of the magnetic field through a 2-sphere surrounding the origin is independent of the radius of the sphere.) Then it must be that

   $$\nabla \cdot \vec{B} = c \; \delta^{(3)}(\vec{x})$$

   for some constant c.[1] Plugging this in the above equation and comparing with the given magnetic flux we find

   $$2\pi m = \int_{S^2} \vec{B} \cdot d\vec{\sigma} = \int_{B^3} \nabla \cdot \vec{B} \; d^3x = c \int_{B^3} \delta^{(3)}(\vec{x}) \; d^3x = c \; .$$

[1] To be precise, derivatives of the delta function would also be allowed, but it's easy to see that they wouldn't reproduce the desired magnetic flux. I will give full marks even if this is not noticed.

2. Using that

$$\nabla \frac{1}{r} = -\frac{\vec{x}}{r^3} , \qquad \Delta \frac{1}{r} = -4\pi \, \delta^{(3)}(\vec{x}) ,$$

where $r = |\vec{x}|$ and $\Delta = \nabla^2$ is the Laplacian, show that

$$\vec{B} = \frac{m}{2} \frac{\vec{x}}{r^3}$$

solves the equation in part 1. [10 marks]

SOLUTION:

We just need to remember that the Laplacian is the divergence of the gradient, or mathematically $\Delta = \nabla \cdot \nabla$. Then

$$\nabla \cdot \left(\frac{m}{2} \frac{\vec{x}}{r^3} \right) = -\frac{m}{2} \nabla \cdot \left(-\frac{\vec{x}}{r^3} \right) = -\frac{m}{2} \nabla \cdot \nabla \frac{1}{r} = -\frac{m}{2} \Delta \frac{1}{r}$$
$$= -\frac{m}{2} (-4\pi) \, \delta^{(3)}(\vec{x}) = 2\pi m \, \delta^{(3)}(\vec{x}) \, .$$

3. Show that the vector potentials $\vec{A}^{\pm}$ for a Dirac monopole, with components

$$A_x^{\pm} = \mp \frac{m}{2} \frac{y}{r(r \pm z)} , \quad A_y^{\pm} = \pm \frac{m}{2} \frac{x}{r(r \pm z)} , \quad A_z^{\pm} = 0$$

satisfy the equations

$$\nabla \times \vec{A}^{\pm} = \frac{m}{2} \frac{\vec{x}}{r^3}$$

in the regions where they are defined. [30 marks]

SOLUTION:

Apologies for the typos in the original version: A_y^{+}, A_z^{+} should have read $A_y^{\pm}, A_z^{\pm}$. I hope that this didn't cause confusion and you looked at the lecture notes instead.

We can do the computation in two ways here: using Cartesian coordinates, with $r = \sqrt{x^2 + y^2 + z^2}$; or using polar coordinates, which make the gauge field easier to write and its field strength easier to calculate, but then we need to switch back to Cartesian coordinates.

Let's use Cartesian coordinates, which is the approach what I expect most of you will have taken. We start by writing

$$A_x^{\epsilon} = -\epsilon \frac{m}{2} \frac{y}{r(r + \epsilon z)} , \qquad A_y^{\epsilon} = \epsilon \frac{m}{2} \frac{x}{r(r + \epsilon z)} , \qquad A_z^{\epsilon} = 0$$

with $\epsilon^2 = 1$, so that we can treat the two cases $\epsilon = \pm 1$ in one go. We'll also use

$$\partial_x r = \frac{x}{r} , \qquad \partial_y r = \frac{y}{r} , \qquad \partial_z r = \frac{z}{r} ,$$

and write r for $\sqrt{x^2 + y^2 + z^2}$.

Next we calculate the derivatives

$$\begin{aligned}
\partial_x A_y^\epsilon &= \epsilon\frac{m}{2}\left[\frac{1}{r(r+\epsilon z)} - x\frac{x/r}{r^2(r+\epsilon z)} - x\frac{x/r}{r(r+\epsilon z)^2}\right] \\
&= \epsilon\frac{m}{2}\frac{1}{r^3(r+\epsilon z)^2}\left[r^2(r+\epsilon z) - x^2(r+\epsilon z) - x^2 r\right] \\
\partial_y A_x^\epsilon &= -\epsilon\frac{m}{2}\frac{1}{r^3(r+\epsilon z)^2}\left[r^2(r+\epsilon z) - y^2(r+\epsilon z) - y^2 r\right] \\
\partial_z A_y^\epsilon &= \epsilon\frac{m}{2}x\left[-\frac{z/r}{r^2(r+\epsilon z)} - \frac{z/r}{r(r+\epsilon z)^2} - \frac{\epsilon}{r(r+\epsilon z)^2}\right] \\
&= -\epsilon\frac{m}{2}\frac{x}{r^3(r+\epsilon z)^2}\left[z(r+\epsilon z) + zr + \epsilon r^2\right] \\
&= -\epsilon\frac{m}{2}\frac{x}{r^3(r+\epsilon z)^2}\epsilon(r+\epsilon z)^2 = -\frac{m}{2}\frac{x}{r^3} \\
\partial_z A_x^\epsilon &= \frac{m}{2}\frac{y}{r^3} \\
\partial_y A_z^\epsilon &= \partial_x A_z = 0\ .
\end{aligned}$$

From this we find the components of $\nabla\times\vec{A}^\epsilon$:

$$\begin{aligned}
(\nabla\times\vec{A}^\epsilon)_x &= \partial_y A_z^\epsilon - \partial_z A_y^\epsilon = \frac{m}{2}\frac{x}{r^3} \\
(\nabla\times\vec{A}^\epsilon)_y &= \partial_z A_x^\epsilon - \partial_x A_z^\epsilon = \frac{m}{2}\frac{y}{r^3} \\
(\nabla\times\vec{A}^\epsilon)_z &= \partial_x A_y^\epsilon - \partial_y A_x^\epsilon \\
&= \epsilon\frac{m}{2}\frac{1}{r^3(r+\epsilon z)^2}\left[(r+\epsilon z)(2r^2 - r^2 + z^2) - r(r^2 - z^2)\right] \\
&= \epsilon\frac{m}{2}\frac{1}{r^3(r+\epsilon z)}\left[r^2 + z^2 - r(r-\epsilon z)\right] \\
&= \epsilon\frac{m}{2}\frac{1}{r^3(r+\epsilon z)}\epsilon z(r+\epsilon z) = \frac{m}{2}\frac{z}{r^3}
\end{aligned}$$

as required.

Ex 27

The Lie algebra $su(2)$ of the group $SU(2)$ has three generators t_1, t_2, t_3 and Lie brackets

$$[t_1, t_2] = it_3 \ , \quad [t_2, t_3] = it_1 \ , \quad [t_3, t_1] = it_2 \ ,$$

where we have fixed the normalization once and for all.

1. Write down all the structure constants ${f_{ab}}^c$ of the Lie algebra $su(2)$. [10 marks]

   SOLUTION:

   Using the definition of the structure constants $[t_a, t_b] = i{f_{ab}}^c t_c$ and the antisymmetry of the Lie bracket (/commutator), we find

   $$ {f_{12}}^3 = {f_{23}}^1 = {f_{31}}^2 = +1 $$
   $$ {f_{21}}^3 = {f_{32}}^1 = {f_{13}}^2 = -1 $$

   and that all other structure constants vanish.

2. Write down generators $t_a^{(\mathbf{2})}$ for the doublet (2-dimensional, or fundamental) representation $\mathbf{2}$, in the above normalization. Calculate the trace $\mathrm{tr}_{\mathbf{2}}(t_a^{(\mathbf{2})} t_b^{(\mathbf{2})})$ and hence the quadratic invariant $C(\mathbf{2})$ of the doublet representation. [20 marks]

   SOLUTION:

   Borrowing results from the previous term, or simply using a basis of 2×2 traceless hermitian matrices, the generators are proportional to the Pauli matrices $t_a^{(\mathbf{2})} = c\sigma_a$, for a constant c. Since the Pauli matrices have the commutators $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$, we need $c = 1/2$ to reproduce the given normalization of the Lie algebra. Hence $t_a^{(\mathbf{2})} = \sigma_a/2$, or explicitly

   $$ t_1^{(\mathbf{2})} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad t_2^{(\mathbf{2})} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad t_3^{(\mathbf{2})} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . $$

   Using the known algebra $\sigma_a\sigma_b = \delta_{ab}1 + i\epsilon_{abc}\sigma_c$, or alternatively by explicit calculation, we find that

   $$ \mathrm{tr}_{\mathbf{2}}(t_a^{(\mathbf{2})} t_b^{(\mathbf{2})}) = \frac{1}{4}\,\mathrm{tr}(\sigma_a\sigma_b) = \frac{1}{2}\delta_{ab} = C(\mathbf{2})\delta_{ab} \ , $$

   hence the quadratic invariant of the doublet is $C(\mathbf{2}) = 1/2$.

3. Write down generators $t_a^{(\mathbf{3})}$ for the triplet (3-dimensional, or adjoint) representation $\mathbf{3}$, in the above normalization. Calculate the trace $\mathrm{tr}_{\mathbf{3}}(t_a^{(\mathbf{3})} t_b^{(\mathbf{3})})$ and hence the quadratic invariant $C(\mathbf{3})$ of the triplet representation. [20 marks]

   SOLUTION:

   There are two ways to approach the first part of the question:

(a) Use the 2 : 1 homomorphism between the groups $SU(2)$ and $SO(3)$, which descends to an isomorphism of Lie algebras, to conclude that the triplet representation of $SU(2)$ is the same as the fundamental (or vector) representation of $SO(3)$, on which $SO(3)$ acts by matrix multiplication, and use previous knowledge about the generators of $so(3)$;

(b) Use the fact that $su(2)$ has dimension 3, and therefore the desired 3-dimensional representation of $SU(2)$ must be the adjoint representation: $\mathbf{3} = \text{adj}$. Then use the general result that the matrix elements of the generators of the adjoint representation are $(t_a^{(\text{adj})})^b{}_c = i f_{ac}{}^b$, and specialize it to the structure constants found in part 1.

Either way, one finds

$$t_1^{(\mathbf{3})} = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \quad t_2^{(\mathbf{3})} = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad t_3^{(\mathbf{3})} = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

which can be checked to obey the Lie brackets of $su(2)$:

$$[t_1^{(\mathbf{3})}, t_2^{(\mathbf{3})}] = i t_3^{(\mathbf{3})} , \quad [t_2^{(\mathbf{3})}, t_3^{(\mathbf{3})}] = i t_1^{(\mathbf{3})} , \quad [t_3^{(\mathbf{3})}, t_1^{(\mathbf{3})}] = i t_2^{(\mathbf{3})} .$$

Then we calculate

$$\text{tr}_{\mathbf{3}}(t_a^{(\mathbf{3})} t_b^{(\mathbf{3})}) = 2\delta_{ab} = C(\mathbf{3})\delta_{ab} ,$$

hence the quadratic invariant of the triplet is $C(\mathbf{3}) = 2$.