Assignment 8

Due date: Friday, 17 March (8pm)

Ex 31

Consider a gauge group G, with Lie algebra $\mathfrak{g}$.

1. Show by explicit calculation that a non-abelian gauge field configuration of the form

$$A_\mu = ih(\partial_\mu h^{-1}) \ ,$$

where $h(x)$ is a space-time dependent element of G, has field strength $F_{\mu\nu} = 0$. [20 marks]

SOLUTION:

We calculate

$$\partial_\mu A_\nu = i(\partial_\mu h)(\partial_\nu h^{-1}) + ih(\partial_\mu \partial_\nu h^{-1}) \ ,$$

therefore

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= i(\partial_\mu h)(\partial_\nu h^{-1}) - i(\partial_\nu h)(\partial_\mu h^{-1}) + ih(\partial_\mu \partial_\nu h^{-1}) - ih(\partial_\nu \partial_\mu h^{-1}) \\ &= i(\partial_\mu h)(\partial_\nu h^{-1}) - i(\partial_\nu h)(\partial_\mu h^{-1}) \ , \end{aligned}$$

where the second derivative terms cancel (as usual, we assume that h^{-1} is sufficiently differentiable so that Schwarz's/Clairaut's theorem applies). The contribution of the commutator is

$$\begin{aligned} -i[A_\mu, A_\nu] &= i[h\partial_\mu h^{-1}, h\partial_\nu h^{-1}] \\ &= ih(\partial_\mu h^{-1})h(\partial_\nu h^{-1}) - ih(\partial_\nu h^{-1})h(\partial_\mu h^{-1}) \ . \end{aligned}$$

Now we use the identity

$$0 = (\partial_\mu \mathbf{1}) = \partial_\mu (hh^{-1}) = (\partial_\mu h)h^{-1} + h(\partial_\mu h^{-1})$$

to get

$$\begin{aligned} -i[A_\mu, A_\nu] &= -i(\partial_\mu h)h^{-1}h(\partial_\nu h^{-1}) + i(\partial_\nu h)h^{-1}h(\partial_\mu h^{-1}) \\ &= -i(\partial_\mu h)(\partial_\nu h^{-1}) + i(\partial_\nu h)(\partial_\mu h^{-1}) \ . \end{aligned}$$

Hence

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0 \ .$$

2. Can you think of a simpler argument to reach the same conclusion? [15 marks]

   SOLUTION:

   Start from a configuration with vanishing gauge field $A_\mu = 0$. The field strength trivially vanishes: $F_{\mu\nu} = 0$. Now perform a gauge transformation with gauge parameter $g = h$. We find that the new (gauge transformed) gauge field A'_μ and field strength $F'_{\mu\nu}$ are

$$\begin{aligned} A'_\mu &= hA_\mu h^{-1} + ih(\partial_\mu h^{-1}) = ih(\partial_\mu h^{-1}) \\ F'_{\mu\nu} &= hF_{\mu\nu}h^{-1} = 0 \ . \end{aligned}$$

   Now, what is primed or unprimed is a matter of point of view: I could have called the primed variables unprimed and vice versa, had I used the inverse gauge transformation. The key point here is that this shows that the field strength of $A_\mu = ih(\partial_\mu h^{-1})$ is $F_{\mu\nu} = 0$. Configurations like $A_\mu = ih(\partial_\mu h^{-1})$, which are obtained by a gauge transformation of the trivial (*i.e.* zero) configuration, are called *pure gauge* configurations.

Ex 34

Consider a gauge theory with Lagrangian density

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \\ &= -\frac{1}{2g_{YM}^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{16\pi^2}\,\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) - (D_\mu\phi)^\dagger D^\mu\phi - V(\phi,\phi^\dagger) \ , \end{aligned}$$

where the scalar potential $V(\phi,\phi^\dagger)$ is a gauge invariant function of ϕ, which transforms in the fundamental representation of the gauge group G, and of $\phi^\dagger = \overline{\phi}^T$.

1. Show that the equations of motion are

$$\begin{aligned} D_\mu D^\mu\phi &= \frac{\partial V}{\partial\phi^\dagger} \\ D_\nu F^{\mu\nu} &= g_{YM}^2 J^\mu \end{aligned}$$

   for a current $J_\mu = J_\mu^a t_a$ that you should find. [35 marks]

   SOLUTION:

   This is admittedly long, but hopefully instructive. Let's start from the equation of motion for the scalar field ϕ. Integrating by parts its kinetic term, we can write the terms in the action involving ϕ as

$$S[\phi, A] = \int d^4x \ \left[\phi^\dagger D_\mu D^\mu\phi - V(\phi,\phi^\dagger) + \dots\,\right] ,$$

where the dots do not involve ϕ. Setting to zero the first variation of the action with respect to $\phi^\dagger$ gives the desired equation of motion

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger} \ .$$

Varying with respect to ϕ gives the hermitian conjugate equation.

The equation of motion for the gauge field A_μ is more challenging. There are several ways to proceed: one is to expand in a basis of generators of the Lie algebra, $A_\mu = A_\mu^a t_a$, use the Lie brackets and the quadratic invariant to write the Lagrangian in terms of A_μ^a, its derivative and the structure constants of the Lie algebra, and then finding the Euler-Lagrange equation for A_μ^a. It is straightforward but tedious, so we'll follow another route. We first note that the theta term is a total derivative term, as in Ex 33, therefore its contribution to the action is a boundary (or surface) term. Its variation is also a boundary term, therefore it does not contribute to the equations of motion.[1] We can thus focus on the dependence of A_μ in the Yang-Mills term and in the kinetic term for ϕ. The kinetic term for ϕ contributes to the action

$$\begin{aligned} S_{\text{matter}}[\phi, A] &= -\int d^4x \ \left(i\phi^\dagger A_\nu \partial^\nu \phi - i\partial^\nu \phi^\dagger A_\nu \phi + \phi^\dagger A^\nu A_\nu \phi + \dots \right) \\ &= -\int d^4x \ \text{tr}\left(iA_\nu(\partial^\nu\phi)\phi^\dagger - iA_\nu\phi(\partial^\nu\phi^\dagger) + A_\beta A^\beta \phi\phi^\dagger + \dots \right) \end{aligned}$$

where the dots don't contain the gauge field. The first line is gauge invariant, so it's equal to its trace, and using cyclicity we obtain the second line. We now take the variation with respect to A_μ and use cyclicity again to find (to first order in the variation δA_μ)

$$\begin{aligned} &S_{\text{matter}}[\phi, A + \delta A] - S_{\text{matter}}[\phi, A] \\ &\qquad = -\int d^4x \ \text{tr}\left((\delta A_\mu)(i(\partial^\mu\phi)\phi^\dagger - i\phi(\partial^\mu\phi^\dagger) + A^\mu\phi\phi^\dagger + \phi\phi^\dagger A^\mu)\right) \\ &\qquad = -\int d^4x \ \text{tr}\left((\delta A_\mu)(i(D^\mu\phi)\phi^\dagger - i\phi(D^\mu\phi)^\dagger)\right) \\ &\qquad \equiv -\int d^4x \ \text{tr}\left((\delta A_\mu)K^\mu\right) \end{aligned}$$

for a current

$$K^\mu = i(D^\mu\phi)\phi^\dagger - i\phi(D^\mu\phi)^\dagger \ .$$

[1] Alternatively, proceed as for the Yang-Mills term below and use the Bianchi identity $D_\mu \tilde{F}^{\mu\nu} = 0$.

For the Yang-Mills term, we write

$$
\begin{aligned}
&- g_{YM}^2 \delta S_{YM}[A] = \int d^4x \ \mathrm{tr}\left((\delta F_{\alpha\beta})F^{\alpha\beta}\right) \\
&= \int d^4x \ \mathrm{tr}\left((\partial_\alpha(\delta A_\beta) - \partial_\beta(\delta A_\alpha) - i(\delta A_\alpha)A_\beta - iA_\alpha(\delta A_\beta) + i(\delta A_\beta)A_\alpha + iA_\beta(\delta A_\alpha))F^{\alpha\beta}\right) \\
&= \int d^4x \ \mathrm{tr}\big(- (\delta A_\beta)\partial_\alpha F^{\alpha\beta} + (\delta A_\alpha)\partial_\beta F^{\alpha\beta} - i(\delta A_\alpha)A_\beta F^{\alpha\beta} - i(\delta A_\beta)F^{\alpha\beta}A_\alpha \\
&\qquad\qquad + i(\delta A_\beta)A_\alpha F^{\alpha\beta} + iA_\beta(\delta A_\alpha)F^{\alpha\beta}A_\beta\big) \\
&= \int d^4x \ \mathrm{tr}\left(- (\delta A_\beta)D_\alpha F^{\alpha\beta} + (\delta A_\alpha)D_\beta F^{\alpha\beta}\right) = 2\int d^4x \ \mathrm{tr}\left((\delta A_\mu)D_\nu F^{\mu\nu}\right)
\end{aligned}
$$

where I integrated by parts, dropped boundary terms and used cyclicity of the trace to go from the second line to the third line, and then used $D_\alpha F^{\alpha\beta} = \partial_\alpha F^{\alpha\beta} - i[A_\alpha, F^{\alpha\beta}]$ to go from the third to the fourth line. In the last equality I relabelled dummy indices and used antisymmetry of $F^{\mu\nu}$. So

$$
S[\phi, A + \delta A] - S[\phi, A] = \int d^4x \ \mathrm{tr}\left((\delta A_\mu)\left(-K^\mu - \frac{2}{g_{YM}^2}D_\nu F^{\mu\nu}\right)\right) \ ,
$$

from which we read off the equation of motion for the gauge field

$$
D_\nu F^{\mu\nu} = -\frac{1}{2}g_{YM}^2 K^\mu \ .
$$

Therefore

$$
J^\mu = -\frac{1}{2}K^\mu = -\frac{i}{2}\left((D^\mu\phi)\phi^\dagger - \phi(D^\mu\phi)^\dagger\right) = \frac{i}{2}\left(\phi(D^\mu\phi)^\dagger - (D^\mu\phi)\phi^\dagger\right) \ .
$$

2. Show that the current J^μ transforms as

$$
J^\mu \mapsto gJ^\mu g^{-1}
$$

under a gauge transformation with group element $g = g(x)$, and that it is covariantly conserved, namely

$$
D_\mu J^\mu = 0 \ .
$$

[30 marks]

SOLUTION:

Either from the explicit form of J^μ in terms of ϕ and A_μ (and using $g^{-1} = g^\dagger$), or using $J^\mu \propto D_\nu F^{\mu\nu}$, we see that J^μ transforms in the adjoint representation of the gauge group:

$$
J^\mu \mapsto gJ^\mu g^{-1} \ .
$$

To show that it's covariantly conserved, again we either calculate explicitly from the above expression for J^μ, and use the equations of motion for $\phi, \phi^\dagger$ as well as the facts that the scalar potential $V(\phi, \phi^\dagger) = U(\phi^\dagger \phi)$ is real and gauge invariant (*i.e.* only dependent on $\phi^\dagger\phi$):

$$\begin{aligned}
D_\mu J^\mu &= \frac{i}{2} D_\mu \left(\phi (D^\mu \phi)^\dagger - (D^\mu \phi)\phi^\dagger\right) \\
&= \frac{i}{2}\left((D_\mu\phi)(D^\mu\phi)^\dagger + \phi(D_\mu D^\mu \phi)^\dagger - (D^\mu\phi)(D_\mu\phi)^\dagger - (D_\mu D^\mu\phi)\phi^\dagger\right) \\
&= \frac{i}{2}\left(\phi(D_\mu D^\mu\phi)^\dagger - (D_\mu D^\mu \phi)\phi^\dagger\right) \\
&= \frac{i}{2}\left(\phi\frac{\partial V}{\partial \phi} - \frac{\partial V}{\partial \phi^\dagger}\phi^\dagger\right) \\
&= \frac{i}{2}\left(\phi\frac{\partial U}{\partial(\phi^\dagger\phi)}\phi^\dagger - \frac{\partial U}{\partial(\phi^\dagger\phi)}\phi\phi^\dagger\right) = 0 \ .
\end{aligned}$$

In the last equality we used that $\frac{\partial U}{\partial(\phi^\dagger\phi)}$ is gauge invariant (*i.e.* a singlet, or one-dimensional representation of the guage group), so it commutes with ϕ.

Alternatively, we can use $J^\mu \propto D_\nu F^{\mu\nu}$, which implies $D_\mu J^\mu \propto D_\mu D_\nu F^{\mu\nu} = \frac{1}{2}[D_\mu, D_\nu]F^{\mu\nu}$. The differential operator $[D_\mu, D_\nu]$ acts on $F^{\mu\nu}$ in the adjoint representation, that is by a commutator. But as a differential operator, $[D_\mu, D_\nu] = iF_{\mu\nu}$, so we get

$$D_\mu J^\mu \propto [F_{\mu\nu}, F^{\mu\nu}] = 0 \ .$$