A CRASH COURSE ON DIFFERENTIAL GEOMETRY

(or how to learn to stop worrying and love changes of coordinates)

- Change of coordinates $x = (x^\mu) \longmapsto x' = (x'^\mu) = \left(x'^\mu(x)\right)$
  on a diff'ble manifold M of $\dim M = n$

From Calculus & AMV:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \longmapsto \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu$$

$$dx^\mu \longmapsto dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

Think of $\{\partial_\mu\}$ as a basis of the tangent space T_pM — point p (w/ coords x)

$\{dx^\mu\}$ " cotangent space T_p^*M.

These are dual vector spaces: $dx^\mu(\partial_\nu) = \delta^\mu_\nu$ by definition.

An element of T_pM is a TANGENT VECTOR $\quad V = V^\mu \partial_\mu \in T_pM$

An element of T_p^*M is a COTANGENT VECTOR / DIFFERENTIAL / DIFFERENTIAL 1-FORM $\quad \alpha = \alpha_\mu dx^\mu \in T_p^*M$

V, α are defined INTRINSICALLY, i.e. they don't care about coordinate changes:

$$V = V^\mu \partial_\mu = V'^\mu \partial'_\mu$$

$$\alpha = \alpha_\mu dx^\mu = \alpha'_\mu dx'^\mu$$

This determines the transfo properties of their components:

$$V^\mu \longmapsto V'^\mu = \frac{\partial x'^\mu}{\partial x^\rho} V^\rho$$

$$\alpha_\mu \longmapsto \alpha'_\mu = \frac{\partial x^\sigma}{\partial x'^\mu}\alpha_\sigma \qquad [\text{* Ex: CHECK!}]$$

NOTE 1. $\alpha(V) = \alpha_\mu dx^\mu(V^\nu \partial_\nu) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \alpha_\mu V^\nu dx^\mu(\partial_\nu) \underset{\substack{\uparrow \\ \text{dual bases}}}{=} \alpha_\mu V^\nu \delta^\mu_\nu = \alpha_\mu V^\mu$.

A (n,k) – TENSOR T (n upper indices, k lower indices) is

$$T \in \underbrace{T_pM \times \ldots \times T_pM}_{n \text{ times}} \times \underbrace{T_p^*M \times \ldots \times T_p^*M}_{k \text{ times}}$$

$$T = T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_k} \, \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_n} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_k}$$

and

$$T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_k} \longmapsto T'^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_k} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \ldots \frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \ldots \frac{\partial x^{\sigma_k}}{\partial x'^{\nu_k}} T^{\rho_1 \ldots \rho_n}{}_{\sigma_1 \ldots \sigma_k}$$

(All of this can be extended to vector/tensor fields by varying the point P. E.g. for a vector field
(and dropping it from TM, T^*M)

$$V = V^\mu(x)\, \partial_\mu = V'^\mu(x')\, \partial'_\mu \qquad \in TM$$

~~"tangent bundle~~ over M"

$$\Rightarrow \quad V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x) .$$
)

• Example: the METRIC TENSOR g is an element of $\mathrm{Sym}^2 T^*M$
$\equiv (T^*M \times T^*M)_{\text{symmetric part}}$

$$g = g_{\mu\nu}(x)\ \tfrac{1}{2}\left(dx^\mu \otimes d\nu^\nu + dx^\nu \otimes dx^\mu\right)$$

Alternatively, declare that $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and write

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu .$$

It takes 2 (tangent) vectors and spits out a number:

$$g(V, W) = g_{\mu\nu}\, dx^\mu(V^\rho \partial_\rho)\, dx^\nu(W^\sigma \partial_\sigma) = g_{\mu\nu} V^\mu W^\sigma .$$

(Dependence on x suppressed)

* Ex : ~~1) Consider the Euclidean metric in $\mathbb{R}^2$ $g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 = \delta_{ij}\, dx^i \otimes dx^j$ and work out how its components change~~

* **Ex:** consider Euclidean $\mathbb{R}^2$ w/ cartesian coords (x^1, x^2).
  1) Work out how the components of a 1-form
  $$A = A_i(x)\,dx^i \qquad \text{(e.g. gauge field, if we ignore gauge transfo's)}$$
  change if you switch to
     a) Polar coordinates (r,θ) s.t. $(x^1 = r\cos\theta,\ x^2 = r\sin\theta)$
     b) Complex coordinates $(z,\bar{z})$ " $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$.
  2) Repeat for a vector field $V = V^i(x)\partial_i$.
  3) Repeat the exercise for the Euclidean metric
  $$g = \delta_{ab}\,dx^a \otimes dx^b = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$
  4) Compare with AMV. (For the metric, look at $\sqrt{\det(g_{\mu\nu})}$)

- Of particular interest are $(0,p)$-tensors which are totally antisymmetric in their indices.

"WEDGE PRODUCT"

Notation : $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \sum_{\sigma = \text{permut. of } p \text{ objects}} (-1)^\sigma\, dx^{\sigma(\mu_1)} \otimes \ldots \otimes dx^{\sigma(\mu_p)}$

e.g. $dx^{\mu_1} \wedge dx^{\mu_2} = (dx^{\mu_1} \otimes dx^{\mu_2} - dx^{\mu_2} \otimes dx^{\mu_1})$.

$\{dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\}$ basis of $\Lambda^p M := \underbrace{(T^*M \times \ldots \times T^*M)}_{p \text{ times}}\Big|_{\text{antisym}}$.

$$\omega = \frac{1}{p!}\,\omega_{\mu_1 \ldots \mu_p}(x)\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \in \Lambda^p M$$

is called a (DIFFERENTIAL) p-FORM

- Example : $F = \frac{1}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$ is a 2-FORM
  ($F_{\mu\nu}$ ← field strength.)

- The EXTERIOR DIFFERENTIAL $d : \Lambda^p M \longrightarrow \Lambda^{p+1} M$ is defined as

$$d\omega = \frac{1}{p!}\,\partial_\mu\, \omega_{\mu_1 \ldots \mu_p}(x)\, dx^\mu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.$$

*** Ex :**

1) Given $A = A_\mu(x)dx^\mu$, show that $F = dA$ has components
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

2) Show that a $U(1)$ gauge transformation of a gauge field A_μ can be written as

$$A \longmapsto A + d\alpha$$

for a smooth function α.

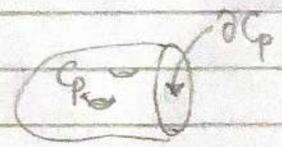
3) Show that $\forall$ smooth $\omega \in \Lambda^p M$,

$$d^2\omega \equiv dd\omega = 0$$

and therefore that $F = dA$ is invariant under $A \mapsto A + d\alpha$.

- Stokes' theorem :

$$\underbrace{\int_{C_p}}_{p\text{-dim'l}} d\alpha_{p-1} = \underbrace{\int_{\partial C_p}}_{\text{boundary of } C_p \ (p-1)\text{-dim'l}} \alpha_{p-1}$$



E.g. $F = dA$ and

$$\underbrace{\int_{\Sigma_2}}_{\text{(2-dim'l) surface with boundary } C_1 = \partial\Sigma_2} F = \int_{\Sigma_2} dA = \int_{\partial\Sigma_2 = C_1} A = \int_{C_1} A,$$

– If $\partial C_p = 0$, then $\int_{C_p} \omega_p = \int_{C_p} (\omega_p + d\alpha_{p-1})$.

– If $d\omega_p = 0$, then $\int_{C_p} \omega_p = \int_{C_p + \partial\Sigma_{p+1}} \omega_p$