SOLUTIONS TO SELECTED EXERCISES

(not in assignments or problems classes)

Ex 1

$$\mathcal{L} = -(\partial_\mu \bar{\Phi})(\partial^\mu \phi) - V(\phi, \bar{\Phi}) \equiv -(\partial_\mu \bar{\Phi})(\partial^\mu \phi) - \lambda(|\phi|^2 - a^2)^2 \qquad \lambda, a > 0$$

$$= |\partial_0 \phi|^2 - |\nabla \phi|^2 - V(\phi, \bar{\Phi})$$

Notation: $|\partial_i \phi|^2 \equiv (\partial_i \bar{\Phi})(\partial_i \phi) = \sum_{i=1}^{3} |\partial_i \phi|^2$

All these notations are fine.

1. $$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \partial_0 \phi + \frac{\partial \mathcal{L}}{\partial \partial_0 \bar{\Phi}} \partial_0 \bar{\Phi} - \mathcal{L} = |\partial_0 \phi|^2 + |\partial_0 \phi|^2 - |\partial_0 \phi|^2 + |\nabla \phi|^2 + V(\phi, \bar{\Phi})$$
$$= |\partial_0 \phi|^2 + |\nabla \phi|^2 + V(\phi, \bar{\Phi})$$

$$\Rightarrow E = \int d^3x \, \mathcal{H} = \int d^3x \, [\underbrace{|\partial_0 \phi|^2}_{\geq 0} + \underbrace{|\nabla \phi|^2}_{\geq 0} + \underbrace{V(\phi, \bar{\Phi})}_{\geq 0}]$$

2. All 3 terms in $\mathcal{H}$ are ≥ 0, so the energy is minimized by setting them to zero individually:

$$\partial_0 \phi = 0, \quad \nabla \phi = 0, \quad \lambda(|\phi|^2 - a^2)^2 = 0$$

so the ground states/vacua are constant field configurations for ϕ which minimize the scalar potential:

$$\{\phi = a e^{i\beta} \mid \beta = \text{const} \sim \beta + 2\pi\} \cong S^1$$

Circle (of radius a) in the complex ϕ plane.

3. Vacuum 1: $\phi = a e^{i\beta_1} \equiv \phi_1$

Vacuum 2: $\phi = a e^{i\beta_2} \equiv \phi_2$

Global U(1) transfo: $\phi \mapsto e^{i\alpha} \phi$

Choose $\alpha = \beta_2 - \beta_1$: $\phi_1 = a e^{i\beta_1} \mapsto e^{i(\beta_2 - \beta_1)} a e^{i\beta_1} = a e^{i\beta_2} = \phi_2$.

Ex 4

Field content: - ϕ cplx scalar, charge 1
- χ cplx scalar, charge 2
- A_μ U(1) gauge field

1. (Gauge invariant) kinetic terms for ϕ, χ:

$$\mathcal{L}_{\text{matter}} = -\left(\partial_\mu\bar{\phi}+iA_\mu\bar{\phi}\right)\left(\partial^\mu\phi-iA^\mu\phi\right)-\left(\partial_\mu\bar{\chi}+2iA_\mu\bar{\chi}\right)\left(\partial^\mu\chi-2iA_\mu\chi\right)$$

2. (Gauge invariant) kinetic term for A_μ:

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \qquad , \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

3. (Gauge invariant) scalar potential of degree ≤ 4:

Gauge invariant terms: $|\phi|^2, |\chi|^2, \underset{\bar{\phi}\phi}{|\phi|^4}, |\chi|^4$
(total charge = 0) $\bar{\chi}\phi^2, \; \bar{\phi}^2\chi$

$$\Rightarrow V = m_\phi^2|\phi|^2 + m_\chi^2|\chi|^2 + \lambda_\phi|\phi|^4 + \lambda_\chi|\chi|^4$$
$$+ \operatorname{Re}\left(a\,\bar{\chi}\phi^2\right)$$

where $m_\phi^2, m_\chi^2, \lambda_\phi, \lambda_\chi$ are real constants, a is a complex constant.

Ex 8

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \qquad (\varepsilon_{0123}=1)$$

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

1.
$$\tilde{F}_{01} = \frac{1}{2}\varepsilon_{0123}F^{23} + \frac{1}{2}\varepsilon_{0132}F^{32} = \varepsilon_{0123}F^{23} = F^{23} = F_{23} = -B_1$$
$$\tilde{F}_{02} = \varepsilon_{0231}F^{31} = -\varepsilon_{0213}F^{31} = \varepsilon_{0123}F^{31} = F^{31} = F_{31} = -B_2$$
$$\tilde{F}_{03} = \varepsilon_{0312}F^{12} = F^{12} = F_{12} = -B_3$$
$$\tilde{F}_{12} = \varepsilon_{1203}F^{03} = F^{03} = -F_{03} = -E_3$$
$$\tilde{F}_{13} = \varepsilon_{1302}F^{02} = -F^{02} = F_{02} = E_2$$
$$\tilde{F}_{23} = \varepsilon_{2301}F^{01} = F^{01} = -F_{01} = -E_1$$

NOTE: $(\vec{E},\vec{B}) \to (-\vec{B},\vec{E})$

$$[\tilde{F}_{\mu\nu}] = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_2 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix}$$

2.
$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_\theta = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$= -\frac{1}{2g^2}\left(-\sum_{i=1}^{3} F_{0i}^2 + \sum_{i<j} F_{ij}^2\right) + \frac{\theta}{8\pi^2}\left(-\sum_{i=1}^{3} F_{0i}\tilde{F}_{0i} + \sum_{i<j} F_{ij}\tilde{F}_{ij}\right)$$

$$= -\frac{1}{2g^2}\left(-\vec{E}^2 + \vec{B}^2\right) + \frac{\theta}{8\pi^2}\left(+\vec{E}\cdot\vec{B} + \vec{E}\cdot\vec{B}\right)$$

$$= \frac{1}{2g^2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{\theta}{4\pi^2}\vec{E}\cdot\vec{B}$$

3.
$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

antisymmetry of $\varepsilon_{\mu\nu\rho\sigma}$
$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(2\partial_\mu A_\nu)(2\partial_\rho A_\sigma) = 2\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu\, \partial_\rho A_\sigma)$$

$$= \partial_\mu\left(2\varepsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right) = \partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right)$$

$$\uparrow$$

$$\underbrace{\varepsilon^{\mu\nu\rho\sigma}}_{\substack{\text{antisymmetric} \\ \text{in } \mu\rho}} A_\nu \underbrace{\partial_\mu\partial_\rho}_{\substack{\text{symmetric} \\ \text{in } \mu\rho}} A_\sigma = 0.$$

Contribution of $\mathcal{L}_\theta$ to EoM:

$$\frac{\partial\mathcal{L}_\theta}{\partial A_\rho} - \partial_\sigma\frac{\partial\mathcal{L}_\theta}{\partial\partial_\sigma A_\rho} = -\partial_\sigma\left(\frac{\partial\mathcal{L}_\theta}{\partial F_{\mu\nu}}\frac{\partial \overbrace{F_{\mu\nu}}^{=\partial_\mu A_\nu - \partial_\nu A_\mu}}{\partial\partial_\sigma A_\rho}\right) = -\partial_\sigma\left(\frac{\theta}{8\pi^2}\tilde{F}^{\mu\nu}\left(\delta^\sigma_\mu\delta^\rho_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right)\right)$$

$$= -\frac{\theta}{4\pi^2}\partial_\sigma\tilde{F}^{\sigma\rho} = -\frac{\theta}{8\pi^2}\varepsilon^{\sigma\rho\mu\nu}\partial_\sigma F_{\mu\nu} = 0$$

by the Bianchi identity (if you don't remember it, sub in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.)

Ex 15

$$N[C] = \frac{1}{2\pi}\oint_C \nabla \arg(\phi)\cdot d\vec{l} = \frac{1}{2\pi}\oint_C d\arg(\phi) = \frac{1}{2\pi}\oint_C \partial_i \arg(\phi)\, dx^i$$

1. $C : [0, 2\pi] \longrightarrow \mathbb{R}^2$
$\tau \longmapsto x(\tau) = (x^1(\tau), x^2(\tau))$ with $x(2\pi) = x(0)$

$$N[C] = \frac{1}{2\pi}\int_0^{2\pi} d\tau\, \dot{x}^i(\tau)\, \partial_i \arg(\phi(x(\tau))) = \frac{1}{2\pi}\Big[\arg\phi(x(\tau))\Big]_{\tau=0}^{\tau=2\pi} = \frac{1}{2\pi}\big(\arg\phi(x(2\pi)) - \arg\phi(x(0))\big)$$

$\phi = |\phi| e^{i\arg(\phi)}$ single valued $\iff \arg\phi \sim \arg\phi + 2\pi$

$x(2\pi) = x(0) \Rightarrow \phi(x(2\pi)) = \phi(x(0)) \iff \arg\phi(x(2\pi)) - \arg\phi(x(0)) = 2\pi n, \quad n \in \mathbb{Z}$

Hence $\quad N[C] = \frac{1}{2\pi} 2\pi n = n \in \mathbb{Z}$.

2. Let D be a disc with boundary the loop C: $\partial D = C$. Then by Stokes' thm

$$N[C] = \frac{1}{2\pi}\oint_C \vec{\nabla}\arg\phi \cdot d\vec{l} = \frac{1}{2\pi}\iint_D (\vec{\nabla}\times\vec{\nabla}\arg\phi)\, d^2x .$$

If $\phi, \frac{1}{\phi} \neq 0$, then $\arg(\phi)$ is (well-)defined, and assuming smoothness $\vec{\nabla}\times\vec{\nabla}\arg\phi = 0$.

3. Using $\arg\phi = \operatorname{Im}\log\phi$,

$$N[C] = \frac{1}{2\pi}\oint_C d\arg\phi = \frac{1}{2\pi}\oint_C d\,\operatorname{Im}\log\phi = \operatorname{Im}\oint_C \frac{d\log\phi}{2\pi} = \operatorname{Re}\oint_C \frac{d\log\phi}{2\pi i}$$

$$\phi \approx c(z-z_0)^n \Rightarrow d\log\phi = n\, d\log(z-z_0) = n\frac{dz}{z-z_0}$$

$$N[C_{z_0}] = \operatorname{Re}\left(n\oint_{C_{z_0}} \frac{dz}{2\pi i(z-z_0)}\right) = \operatorname{Re}(n) = n \qquad \text{by Cauchy's thm.}$$

(Alternatively/explicitly, parametrize C_{z_0} by $z(\tau) = z_0 + \varepsilon e^{i\tau}$ where $0 < \varepsilon \ll 1$, $\tau \sim \tau + 2\pi$:

$$\oint_{C_{z_0}} \frac{dz}{2\pi i(z-z_0)} = \int_0^{2\pi} \frac{d\tau\, z'(\tau)}{2\pi i\cdot\varepsilon e^{i\tau}} = \frac{1}{2\pi i\cancel{\varepsilon}}\int_0^{2\pi} d\tau \frac{\cancel{\varepsilon} i \cancel{e^{i\tau}}}{\cancel{e^{i\tau}}} = \frac{1}{2\pi}\cdot 2\pi = 1 .$$
)

4. $\phi \approx c(z-z_0)^n(\bar{z}-\bar{z}_0)^m \Rightarrow d\log\phi = n\, d\log(z-z_0) + m\, d\log(\bar{z}-\bar{z}_0)$

$$\Rightarrow d\,\mathrm{Im}\log\phi = n\,\frac{d\log(z-z_0) - d\log(\bar{z}-\bar{z}_0)}{2i} + m\,\frac{d\log(\bar{z}-\bar{z}_0) - d\log(z-z_0)}{2i}$$

$$= (n-m)\, d\,\mathrm{Im}\log(z-z_0)$$

Therefore, by the same logic as in part 3,

$$N[C_{z_0}] = n-m.$$

Ex 16

$$\phi = \phi(\underline{x}) \qquad \nabla^2\phi - \lambda(|\phi|^2 - v^2)\phi = 0$$

$$\underline{x} = (x^1, x^2)$$

1. $x^1 + ix^2 = re^{i\theta}$ $\quad(r,\theta)$ polar coord's on $\mathbb{R}^2$

Ansatz $\quad \phi = f(r)\, e^{i\theta}$

$$N = \frac{1}{2\pi}\oint_{S^1_\infty} \partial_i(\arg\phi)\, dx^i = \frac{1}{2\pi}\oint_{S^1_\infty} \partial_i\theta\, dx^i = \frac{1}{2\pi}\oint_{S^1_\infty}\left(\frac{\partial\theta}{\partial r}dr + \frac{\partial\theta}{\partial\theta}d\theta\right) = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

where we used that the differential $d(\ldots) = \partial_i(\ldots)\,dx^i$ is independent of the choice of coordinates and that $\oint_{S^1_\infty} \frac{\partial\theta}{\partial r}dr = 0$ because $\frac{\partial\theta}{\partial r} = 0$ (and also because the integral is at fixed r).

2. Using the change of coordinates from Cartesian (x^1, x^2) to polar (r,θ) as in Ex 34 one can find (see Calculus or AMV)

$$\nabla^2 = \partial^2_{x^1} + \partial^2_{x^2} = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_\theta$$

[In an exam, I would ask you to derive the result as a part of the question, or alternatively I'd provide the formula, depending on the length of the question.]

Next we note that $\partial_\theta^2 e^{i\theta} = -e^{i\theta}$. So

$$\nabla^2 \left(f(r) e^{i\theta}\right) = e^{i\theta}\left(f''(r) + \frac{1}{r} f'(r) - \frac{1}{r^2} f(r)\right). \qquad (1)$$

$$\lambda\left(v^2 - |\phi|^2\right)\phi \underset{\substack{\uparrow \\ \phi = f e^{i\theta}}}{=} \lambda\left(v^2 - f^2\right) f e^{i\theta} \qquad (2)$$

$$e^{-i\theta}\left((1)+(2)\right) = 0 \iff f'' + \frac{1}{r} f' - \frac{1}{r^2} f + \lambda\left(v^2 - f^2\right) f = 0.$$

Ex 17

$$E = \int d^2x\, \mathcal{E} = \int d^2x \left[|\nabla\phi|^2 + \frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2\right]$$

1. $\phi = \rho e^{i\alpha}$: $d\phi = d\rho \cdot e^{i\alpha} + \rho\, d(e^{i\alpha}) = e^{i\alpha}(d\rho + i\rho\, d\alpha)$

$$\Rightarrow |d\phi|^2 = |e^{i\alpha}(d\rho + i\rho\, d\alpha)|^2 = (d\rho)^2 + \rho^2 (d\alpha)^2.$$

$$\Rightarrow |\nabla\phi|^2 = (\nabla\rho)^2 + \rho^2 (\nabla\alpha)^2$$ using $df = \vec{\nabla} f \cdot d\vec{x}$ (alternatively, replace $d \to \nabla$ above)

$$\Rightarrow E = \int d^2x \left[(\nabla\rho)^2 + \rho^2(\nabla\alpha)^2 + \frac{\lambda}{2}\left(\rho^2 - v^2\right)^2\right].$$

2. $(x^1, x^2) \to (r, \theta)$ polar coordinates.

Calculus/AMV : $\nabla g(r,\theta) = \frac{\partial g}{\partial r} \underline{e}_r + \frac{1}{r}\frac{\partial g}{\partial \theta} \underline{e}_\theta \Rightarrow (\nabla g)^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \theta}\right)^2.$

If $g = \theta$: $(\nabla\theta)^2 = \frac{1}{r^2}$.

$\phi = f(r) e^{i\theta} \iff \rho = f(r),\ \alpha = \theta$, so $\rho^2 (\nabla\alpha)^2 = f^2 (\nabla\theta)^2 = \frac{f^2}{r^2}$.

Hence the contribution to the energy of the $\int d^2x\, \rho^2 (\nabla\alpha)^2$ term for $R_0 < r < R$ is

$$\int_0^{2\pi} d\theta \int_{R_0}^{R} dr \cdot r \frac{f^2(r)}{r^2} = 2\pi \int_{R_0}^{R} dr \frac{f^2(r)}{r} \underset{\substack{\uparrow \\ \text{Assume } 1 \ll R_0 < R}}{\approx} 2\pi v^2 \int_{R_0}^{R} \frac{dr}{r} = 2\pi v^2 \log\left(\frac{R}{R_0}\right) \xrightarrow[\substack{R\to\infty \\ (\text{fixed } R_0)}]{} \infty.$$

Ex 18

$$\mathcal{L} = -\frac{1}{2} G_{ab}(\phi)\, \partial_\mu \phi^a \partial^\mu \phi^b - V(\phi)$$

$$= \frac{1}{2} G_{ab}(\phi)(\partial_0\phi^a)(\partial_0\phi^b) \qquad \equiv \mathcal{T}$$

$$-\frac{1}{2} G_{ab}(\phi)(\partial_i\phi^a)(\partial_i\phi^b) \qquad \equiv -\mathcal{E}_K$$

$$- V(\phi) \qquad \equiv -\mathcal{E}_V \equiv -V$$

1. Euler–Lagrange eqns (from $\delta S = \delta \int d^{D+1}x\, \mathcal{L} = 0$ for all $\delta\phi$):

$$0 = \frac{\partial \mathcal{L}}{\partial \phi^c} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^c}$$

For static field configurations $(\partial_0 \phi^a = 0)$: $\mathcal{T}, \frac{\partial \mathcal{T}}{\partial \phi}, \frac{\partial \mathcal{T}}{\partial \partial_0 \phi} = 0$.

Hence the E-L eqns become

$$0 = \frac{\partial}{\partial \phi^c}(\cancel{\mathcal{T}} - \mathcal{E}_K - V) - \underbrace{\partial_0 \frac{\partial}{\partial(\partial_0\phi^c)}(\cancel{\mathcal{T}} - \mathcal{E}_K - V)}_{0} - \partial_i \frac{\partial}{\partial(\partial_i \phi^c)}(\cancel{\mathcal{T}} - \mathcal{E}_K - V)$$

$$= -\left[\frac{\partial}{\partial\phi^c}(\mathcal{E}_K + V) - \partial_i \frac{\partial}{\partial(\partial_i\phi^c)}(\mathcal{E}_K + V)\right] = -\left[\frac{\partial \mathcal{E}}{\partial \phi^c} - \partial_i \frac{\partial \mathcal{E}}{\partial(\partial_i \phi^c)}\right]$$

where $\mathcal{E} = \mathcal{E}_K + V$ is the static energy density. These are precisely the equations that are obtained by requiring that $E[\phi] = \int d^D x\, (\mathcal{E}_K + V)$ is stationary for all $\delta\phi$. Indeed, recall the derivation of

$$0 = \delta S[\phi(x^0, \vec{x})] = \delta \int d^{D+1}x\, \mathcal{L}(\phi, \partial_\mu \phi) \quad \forall \delta\phi \quad \Rightarrow \quad \text{E-L eqns} \quad 0 = \frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} .$$

By the same maths, replacing $\phi(x^0, \vec{x})$ by $\phi(\vec{x})$ and S by E we get

$$0 = \delta E[\phi(\vec{x})] = \delta \int d^D x\, \mathcal{E}(\phi, \partial_i \phi) \quad \forall \delta\phi \quad \rightarrow \quad 0 = \frac{\partial \mathcal{E}}{\partial \phi^a} - \partial_i \frac{\partial \mathcal{E}}{\partial \partial_i \phi^a} .$$

2. Use x instead of $\vec{x}$ for ease of notation. Let $\phi_1(x)$ be a static sol'n of the EoM found in part 1. Consider the 1-parameter family of field configurations

$$\phi(x) = \phi_\lambda(x) \equiv \phi_1(\lambda x) .$$

$$E_V[\phi_\lambda] = \int d^D x \; V(\phi_\lambda(x)) = \int d^D x \; V(\phi_1(\lambda x)) \underset{y=\lambda x}{\overset{\uparrow}{=}} \lambda^{-D} \int d^D y \; V(\phi_1(y)) = \lambda^{-D} E_V[\phi_1]$$

$$E_K[\phi_\lambda] = \int d^D x \; \tfrac{1}{2} G_{ab}(\phi_\lambda(x)) (\partial_i \phi_\lambda^a(x)) (\partial_i \phi_\lambda^b(x)) = \int \underbrace{d^D x}_{\lambda^{-D} d^D(\lambda x)} \; \tfrac{1}{2} G_{ab}(\phi_1(\lambda x)) \underbrace{\left(\frac{\partial}{\partial x^i} \phi_1^a(\lambda x)\right)}_{\lambda \frac{\partial}{\partial(\lambda x^i)} \phi_1^a(\lambda x)} \underbrace{\left(\frac{\partial}{\partial x^i} \phi_1^b(\lambda x)\right)}_{\lambda \frac{\partial}{\partial(\lambda x^i)} \phi_1^b(\lambda x)}$$

$$\underset{y=\lambda x}{\overset{\uparrow}{=}} \lambda^{2-D} \int d^D y \; \tfrac{1}{2} G_{ab}(\phi_1(y)) \left(\frac{\partial}{\partial y^i} \phi_1^a(y)\right) \left(\frac{\partial}{\partial y^i} \phi_1^b(y)\right)$$

$$= \lambda^{2-D} E_K[\phi_1]$$

$$\Longrightarrow \quad E_K[\phi_\lambda] + E_V[\phi_\lambda] = \lambda^{2-D} E_K[\phi_1] + \lambda^{-D} E_V[\phi_1] .$$

Ex 19

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \overline{D_\mu \phi} \, D^\mu \phi - \underbrace{\frac{\lambda}{2} (|\phi|^2 - v^2)^2}_{= U(|\phi|^2)}$$

1. EoM : see Ex 10. The result is

$$0 = D_\mu D^\mu \phi - U'(|\phi|^2) \phi = D_\mu D^\mu \phi - \lambda (|\phi|^2 - v^2) \phi$$

$$\partial_\mu F^\mu{}_\nu = g^2 J_\nu = g^2 \cdot i (\bar\phi D_\nu \phi - \overline{D_\nu \phi} \cdot \phi) = g^2 \cdot i (\bar\phi \partial_\nu \phi - \overline{\partial_\nu \phi} \cdot \phi) + 2 g^2 A_\nu |\phi|^2 .$$

2. Gauge $A_0 = 0$, static solutions of the form

$$\phi(\underline{x}) = v e^{i\theta} f(vr)$$

$$A_j(\underline{x}) = \varepsilon_{jk} \hat{x}^k \frac{a(vr)}{r} .$$

$$\underline{x} = (x^1, x^2) = (r\cos\theta, r\sin\theta)$$

$(\hat{x}^1, \hat{x}^2) = (\cos\theta, \sin\theta)$

$$A_r = \frac{\partial x^1}{\partial r} A_1 + \frac{\partial x^2}{\partial r} A_2 = \cos\theta \, A_1 + \sin\theta \, A_2 = \cos\theta \sin\theta \frac{a(vr)}{r} + \sin\theta (-\cos\theta) \frac{a(vr)}{r} = 0 .$$

$$A_\theta = \frac{\partial x^1}{\partial \theta} A_1 + \frac{\partial x^2}{\partial \theta} A_2 = r(-\sin\theta \, A_1 + \cos\theta \, A_2) = r(-\sin^2\theta - \cos^2\theta) \frac{a(vr)}{r} = -a(vr) .$$

$$\Rightarrow F_{r\theta} = -v a'(vr)$$

Calculus/AMV for Laplacian (then $\partial_\mu \to D_\mu$):

$$D^2 = D_i D_i = D_r^2 + \frac{1}{r} D_r + \frac{1}{r^2} D_\theta^2 \underset{\substack{\uparrow \\ A_r=0 \\ A_\theta = -a(vr)}}{=} \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2} \underbrace{(\partial_\theta + i a(vr))^2}_{= \partial_\theta^2 + 2ia(vr)\partial_\theta - a^2(vr)}$$

acting on a field of charge 1.

Acting on $\phi = v e^{i\theta} f(vr) \equiv v e^{i\theta} f(u)$:

$$\partial_r^2 \left(v e^{i\theta} f(vr)\right) = v^3 e^{i\theta} f''(vr) = v^3 e^{i\theta} f''(u)$$

$$\frac{1}{r} \partial_r \left(v e^{i\theta} f(vr)\right) = v^2 e^{i\theta} \frac{f'(vr)}{r} = v^3 e^{i\theta} \frac{f'(u)}{u}$$

$$\frac{1}{r^2}(\partial_\theta + i a(vr))^2 \left(v e^{i\theta} f(vr)\right) = -v e^{i\theta} \frac{f(vr)}{r^2} - 2v e^{i\theta} a(vr) \frac{f(vr)}{r^2} - v e^{i\theta} \frac{a^2(vr) f(vr)}{r^2}$$

$$= -\frac{v^3 e^{i\theta}}{u^2} f(u) \left(1 + a(u)\right)^2$$

$$-\lambda \left(|\phi|^2 - v^2\right)\phi = -\lambda v^3 e^{i\theta} \left(f^2(vr) - 1\right) f(vr) = -\lambda v^3 e^{i\theta} \left(f^2(u) - 1\right) f(u)$$

So the ϕ EoM is:

$$f''(u) + \frac{1}{u} f'(u) - \frac{1}{u^2} (1 + a(u)) f(u) + \lambda \left(1 - f^2(u)\right) f(u) = 0 . \qquad (*)$$

- The A_μ EoM is trivially satisfied $(0=0)$ for $\underline{\nu = 0}$.

For $\underline{\nu = j}$ we have $\partial_i F^i{}_j = g^2 J_j \equiv g^2 i (\bar\phi \partial_j \phi - \overline{\partial_j \phi}\, \phi) + 2g^2 A_j |\phi|^2$ and we can either use Cartesian coordinates and then switch to polar coordinates, or use polar coordinates directly. I'll follow the latter approach. We need to remember that in polar coordinates $[g_{\mu\nu}] = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, so $[g^{\mu\nu}] = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$.

Flat metric

Hence

$$F^r{}_\theta = g^{rr} F_{r\theta} = F_{r\theta} = -v\, a'(vr), \qquad F^\theta{}_r = g^{\theta\theta} F_{\theta r} = r^{-2} F_{\theta r} = v\, r^{-2} a'(vr)$$

$\boxed{j=r}$ $\quad \partial_\theta F^\theta{}_r = 0, \quad J_r \propto \operatorname{Im}(\bar\phi \partial_r \phi) = 0 .$ ✓

$\boxed{j=\theta}$ $\quad \partial_r F^r{}_\theta = -v^2 a''(vr), \quad J_\theta = -2g^2 |\phi|^2 - 2g^2 a(vr) |\phi|^2 = -2g^2 (1 + a(vr)) v^2 f^2(vr)$

So the eqn is

$$a''(u) = 2g^2 (1 + a(u)) f^2(u) \qquad (**)$$

The system of ODEs is $\left\{ \begin{array}{l} (*) \\ (**) \end{array} \right.$

Ex 23

1. $K_{ab} = \mathrm{tr}(t_a t_b)\,, \qquad f_{abc} := f_{ab}{}^d K_{dc}$

$$\mathrm{tr}([t_a,t_b]t_c) = i f_{ab}{}^d\, \mathrm{tr}(t_d t_c) = i f_{ab}{}^d K_{dc} \equiv i f_{abc}$$

$$\begin{array}{c} \| \\ \mathrm{tr}(t_a t_b t_c - t_b t_a t_c) \underset{\substack{\uparrow \\ \text{cyclicity of tr}}}{=} \mathrm{tr}(t_b t_c t_a - t_b t_a t_c) = -\mathrm{tr}(t_b[t_a,t_c]) = -i f_{acb} \end{array}$$

$$\text{So}\quad \begin{array}{ccccc} f_{abc} & = - & f_{acb} & & f_{bca} \\ \| & & \| & & \| \\ -f_{bac} & & f_{cab} & = & -f_{cba} \end{array}$$

(vertical equalities follow from $f_{ab}{}^d = -f_{ba}{}^d$)

2. $[t_a, t_b] = i f_{ab}{}^c t_c$

$\left[\#(f_{abc}) : \quad \frac{d(d-1)(d-2)}{6}\right]$

$$\Rightarrow 0 = \mathrm{tr}_{\underline{r}}\left(t_a^{(\underline{r})} t_b^{(\underline{r})} - t_a^{(\underline{r})} t_b^{(\underline{r})}\right) \underset{\substack{\uparrow \\ \text{ciclicity}}}{=} \mathrm{tr}_{\underline{r}}\left(t_a^{(\underline{r})} t_b^{(\underline{r})} - t_b^{(\underline{r})} t_a^{(\underline{r})}\right) = i f_{ab}{}^c\, \underset{\substack{\# \\ 0\ \forall a}}{\mathrm{tr}_{\underline{r}}\left(t_c^{(\underline{r})}\right)} \quad \underset{a,b}{\forall}$$

There are more linear independent combinations of $\mathrm{tr}_{\underline{r}}(t_c^{(\underline{r})})$ which vanish than there are $t_c^{(\underline{r})}$, so $\mathrm{tr}_{\underline{r}}(t_c^{(\underline{r})}) = 0 \quad \forall c.$

Ex 24

1. $t_a^{(\bar{\underline{r}})} := -\left(t_a^{(\underline{r})}\right)^T$

$$\left[t_a^{(\underline{r})}, t_b^{(\underline{r})}\right] = i f_{ab}{}^c\, t_c^{(\underline{r})}$$

We need to check that $t_a^{(\bar{\underline{r}})}$ obey the same Lie brackets:

$$\left[-t_a^{(\underline{r})T}, -t_b^{(\underline{r})T}\right] = \left[t_b^{(\underline{r})}, t_a^{(\underline{r})}\right]^T = -\left[t_a^{(\underline{r})}, t_b^{(\underline{r})}\right]^T = -i f_{ab}{}^c \left(t_c^{(\underline{r})}\right)^T$$

$$= i f_{ab}{}^c \left(-t_c^{(\underline{r})}\right)^T$$

2. $$\phi \longmapsto e^{i\alpha^a t_a^{(r)}}\phi$$

$$\Rightarrow \quad \phi^* \longmapsto e^{-i\alpha^a t_a^{(r)*}}\phi^* = e^{-i\alpha^a t_a^{(r)T}}\phi^*$$

where we used $\alpha^a \in \mathbb{R}$ and $t_a^{(r)\dagger} = t_a^{(r)}$, which holds for representations of compact Lie groups.

3. $$\phi^\dagger \longmapsto \phi^\dagger e^{-i\alpha^a t_a^{(r)}}$$

$$\Rightarrow \quad \phi^\dagger\phi \longmapsto \phi^\dagger e^{-i\alpha^a t_a^{(r)}} e^{i\alpha^a t_a^{(r)}}\phi = \phi^\dagger\phi \, .$$

$\boxed{\text{Ex 25}}$

$$\mathrm{ad}_x(y) = [x,y] \quad , \qquad \left(t_a^{(\mathrm{adj})}\right)^b{}_c = i f_{ac}{}^b$$

1. $$\mathrm{ad}_{t_a}(y^b t_b) \underset{\substack{\uparrow \\ \text{linearity}}}{=} y^b\, \mathrm{ad}_{t_a}(t_b) = y^b [t_a, t_b] = i f_{ab}{}^c y^b t_c = \left(t_a^{(\mathrm{adj})}\right)^c{}_b y^b t_c$$
$$= \left(t_a^{(\mathrm{adj})}\right)^b{}_c y^c t_b$$

2. $$K_{ab} = \mathrm{tr}_{\mathrm{adj}}\left(t_a^{(\mathrm{adj})} t_b^{(\mathrm{adj})}\right) = \left(t_a^{(\mathrm{adj})}\right)^c{}_d \left(t_b^{(\mathrm{adj})}\right)^d{}_c = i f_{ad}{}^c \cdot i f_{bc}{}^d$$
$$= - f_{ad}{}^c f_{bc}{}^d = f_{ad}{}^c f_{cb}{}^d$$

3. $$C(\mathrm{adj})\, \delta_{ab} = K_{ab} = f_{ad}{}^c f_{cb}{}^d$$

$\delta^{ba} \times \nearrow$: $$C(\mathrm{adj}) \underset{\substack{\| \\ \dim \mathfrak{g}}}{\delta^a_a} = \delta^{ab} f_{ad}{}^c f_{cb}{}^d = -\delta^{ab} f_{ad}{}^c f_{bc}{}^d = -\delta^{ab} f_{adc} f_b{}^{cd}$$
$$= \delta^{ab} f_{acd} f_b{}^{cd}$$

$$\Rightarrow \quad C(\mathrm{adj}) = \frac{\delta^{ab} f_{acd} f_b{}^{cd}}{\dim \mathfrak{g}}$$

Ex 36

$$\mathcal{L} = -\frac{1}{2g_{YM}^2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \mathrm{tr}\left(D_\mu\phi\, D^\mu\phi\right) - V(\phi) \quad , \qquad V(\phi) = \lambda\left(\tfrac{1}{2}\mathrm{tr}(\phi^2) - v^2\right)^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]$$

$$D_\mu\phi = \partial_\mu\phi - i\left[A_\mu, \phi\right]$$

1. We need to calculate the Hamiltonian density (or energy density)

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\partial_0\phi^a{}_b}\,\phi^a{}_b + \frac{\partial\mathcal{L}}{\partial\partial_0(A_\mu)^a{}_b}\,(A_\mu)^a{}_b - \mathcal{L}$$

and then specialize to static field configurations in a gauge where $A_0 = 0$. It may look as though we need to calculate derivatives of $\mathcal{L}$ and worry about lots of a, b indices, but luckily for us

$$\frac{\partial\mathcal{L}}{\partial\partial_0\phi^a{}_b} = -(D^0\phi)^b{}_a = (D_0\phi)^b{}_a \;\; = 0 \qquad \text{if } \;\partial_0\phi = A_0 = 0$$

$$\frac{\partial\mathcal{L}}{\partial\partial_0 A_0} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\partial_0(A_i)^a{}_b} = \frac{2}{g_{YM}^2}\,(F_{0i})^b{}_a \;\; = 0 \qquad \text{if } \;\partial_0 A_i = A_0 = 0\,.$$

(There's no need to be careful about indices and numerical factors to reach this conclusion.)

So, if we restrict to static field configurations in the $A_0 = 0$ gauge, we simply find

$$\mathcal{H} = -\mathcal{L}\Big|_{\partial_0 = A_0 = 0} = \frac{1}{2g_{YM}^2}\,\mathrm{tr}\left(F_{ij}F_{ij}\right) + \mathrm{tr}\left(D_i\phi\, D_i\phi\right) + V(\phi)$$

where all indices i,j are spatial (and summed over) $i,j \in \{1,2,3\}$. Next we use $F_{ij} = -\varepsilon_{ijk}B_k$, so

$$F_{ij}F_{ij} = \underbrace{\varepsilon_{ijk}\,\varepsilon_{ij\ell}}\, B_k B_\ell = 2\,\delta_{k\ell}$$

$= 2\delta_{k\ell}$, because k must be equal to ℓ or the result is zero, and the factor of 2 is because there are two options. E.g. if $k = \ell = 3$ we have $(i,j) \in \{(1,2), (2,1)\}$.

So

$$E = \int d^3x \left[\frac{1}{g_{YM}^2}\,\mathrm{tr}(B_k B_k) + \mathrm{tr}\left(D_i\phi\, D_i\phi\right) + V(\phi)\right]$$

The energy / Hamiltonian density is a sum of squares, so it's minimized by setting all squares to zero separately :

$$B_k = 0 \;\; \forall k, \quad D_i \phi = 0 \;\; \forall i, \quad \frac{1}{2}\,\mathrm{tr}(\phi^2) = v^2$$

$$\Updownarrow$$

$$F_{ij} = 0 \;\; \forall i,j$$

$F_{ij} = 0$ means that $A_i(\underline{x})$ is pure gauge, so by a gauge transformation we can set $A_i = 0$. Then $D_i = \partial_i$, so the second eqn sets ϕ to be a constant, and letting $\phi = \phi^a \sigma_a$ where σ_a are Pauli matrices, the third eqn becomes

$$2v^2 = \mathrm{tr}\left(\phi^a \sigma_a \phi^b \sigma_b\right) = \phi^a \phi^b\, \mathrm{tr}(\sigma_a \sigma_b) = \phi^a \phi^b \cdot 2\delta_{ab} = 2\left((\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2\right)$$

$$\Longleftrightarrow \quad \sum_{a=1}^{3} (\phi^a)^2 = v^2 .$$

2. ϕ is a traceless hermitian 2×2 matrix, since it belongs to the Lie algebra of $G = SU(2)$. Linear algebra tells us that we can diagonalize ϕ by a unitary transformation $U \in U(2)$:

$$\phi = U^{-1} \Lambda U ,$$

not to be confused with the scalar potential

where $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \lambda \sigma_3$. We can write $\underset{U(2)}{U} = \underset{U(1)}{e^{i\beta}}\, \underset{SU(2)}{V}$, where $V \in SU(2)$. Then

$$\phi = V^{-1} \Lambda V ,$$

since $e^{-i\beta} e^{i\beta} = 1$.

Imposing the vacuum eqns we find that $\lambda = \text{const}$ and

$$2v^2 = \mathrm{tr}(\phi^2) = \mathrm{tr}(\Lambda^2) = 2\lambda^2 \quad \Longleftrightarrow \quad \lambda = \pm v .$$

So we seem to conclude that

$$\phi = \pm V^{-1} \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} V .$$

But we can do better. Let $V = \tilde{V} \cdot W$, where $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$. Then

$$W^{-1} \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} W = \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix},$$

so conjugation by W flips sign to v.

Since both $\tilde{V}, W \in SU(2)$, they can both be absorbed (or undone) by an $SU(2)$ gauge transformation. Hence we conclude that up to a gauge transformation

$$\phi = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} = v\sigma_3$$

for vacua.

In order to see that this preserves an $H = U(1)$ subgroup of the gauge group $G = SU(2)$, we note that if we take

$$V = u \equiv e^{i\beta\sigma_3} = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix},$$

which defines a $U(1)$ subgroup of $SU(2)$, then

$$\phi = v\sigma_3 \longmapsto u^{-1}\phi u = v\, \underbrace{e^{-i\beta\sigma_3}\sigma_3 e^{i\beta\sigma_3}}_{\text{they all commute!}} = v\, e^{-i\beta\sigma_3} e^{i\beta\sigma_3}\sigma_3 = v\sigma_3 = \phi$$

or more explicitly

$$\phi = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \longmapsto u^{-1}\phi u = \begin{pmatrix} e^{-i\beta} & 0 \\ 0 & e^{i\beta} \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}\begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} = \phi .$$