SOLUTIONS TO SELECTED EXERCISES

(not in assignments or problems classes)

Ex 1

1. Start from $F^{0i} = -F^{i0} = E_i$, $F^{ij} = \varepsilon_{ijk} B_k$ $\left(\begin{matrix} F_{12} = B_3 \\ F_{23} = B_1 \\ F_{31} = B_2 \end{matrix}\right)$

and lower indices with $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$. Since η is diagonal, we just pick a $-$ sign when lowering a 0 index, and a $+$ sign when lowering an $i=1,2,3$ index.

$$F_{0i} = \eta_{00}\eta_{ii} F^{0i} = -F^{0i} = -E_i \qquad \text{(no summation of indices)}$$

$$F_{ij} = \eta_{ii}\eta_{jj} F^{ij} = F_{ij} = \varepsilon_{ijk} B_k \qquad (\quad \text{"} \quad)$$

so

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

2. Similarly, we start from $\varepsilon^{0123} = 1$ to get

$$\varepsilon_{0123} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\,\varepsilon^{0123} = -\varepsilon^{0123} = -1 .$$

The same computation applies to any other non-vanishing component of ε, since it must have exactly one time and 3 space indices:

$$\varepsilon_{\sigma(0)\sigma(1)\sigma(2)\sigma(3)} = -\varepsilon^{\sigma(0)\sigma(1)\sigma(2)\sigma(3)} = -\,\mathrm{sign}(\sigma) .$$

3.

$$\partial_\nu F^{\mu\nu} = J^\mu \qquad\qquad \varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\mu=0: \quad \partial_i F^{i0} = J^0 \Leftrightarrow \underbrace{\partial_i E_i}_{\vec\nabla\cdot\vec E} = \rho$$

$$\mu=i: \quad \partial_0 F^{i0} + \partial_j F^{ij} = J^i$$

$$\Updownarrow$$

$$-\partial_t E_i + \varepsilon_{ijk}\,\partial_j B_k = j_i$$

$$\Updownarrow$$

$$-\partial_t \vec E + \vec\nabla\times\vec B = \vec j$$

$$\vec\nabla\cdot\vec B = 0$$

$$\Updownarrow$$

$$\mu=0: \quad 0 = \varepsilon^{0ijk}\,\partial_i F_{jk} = \partial_i\left(\varepsilon_{ijk}F_{jk}\right) = 2\partial_i B_i$$

$$\mu=i: \quad 0 = \varepsilon^{i0jk}\,\partial_0 F_{jk} + \varepsilon^{ij0k}\,\partial_j F_{0k} + \varepsilon^{ijk0}\partial_j F_{k0}$$

$$= -\partial_0(\varepsilon_{ijk}F_{jk}) + \varepsilon_{ijk}\left(-\partial_j E_k - \partial_j E_k\right)$$

$$= -2\,\partial_0 B_i - 2\,\varepsilon_{ijk}\,\partial_j E_k$$

$$\Updownarrow$$

$$\vec 0 = \partial_t \vec B + \vec\nabla\times\vec E$$

4.

$$F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}$$

$$= -2\sum_i F_{0i}^{\,2} + 2\sum_{i<j} F_{ij}^{\,2} = -2\sum_i E_i^2 + 2\sum_k B_k^2$$

$$= -2\left(\vec E^2 - \vec B^2\right).$$

5.

$$\rho = J^0 \longmapsto \rho' = J'^0 = \Lambda^0{}_\mu J^\mu = \Lambda^0{}_0 J^0 + \Lambda^0{}_1 J^1 = \cosh\lambda\cdot\rho + \sinh\lambda\cdot j_1$$

$$j_1 = J^1 \longmapsto j_1' = J'^1 = \Lambda^1{}_\mu J^\mu = \Lambda^1{}_0 J^0 + \Lambda^1{}_1 J^1 = \sinh\lambda\cdot\rho + \cosh\lambda\cdot j_1$$

$$j_{2,3} \longmapsto j'_{2,3} = j_{2,3}.$$

$$E_1 = F^{01} \longmapsto E_1' = F'^{01} = \Lambda^0{}_\mu\Lambda^1{}_\nu F^{\mu\nu} = \Lambda^0{}_0\Lambda^1{}_1 F^{01} + \Lambda^0{}_1\Lambda^1{}_0 F^{10} = \left(\cosh^2\lambda - \sinh^2\lambda\right)E_1 = E_1$$

$$E_2 = F^{02} \longmapsto E_2' = F'^{02} = \Lambda^0{}_\mu\Lambda^2{}_\nu F^{\mu\nu} = \Lambda^0{}_0\Lambda^2{}_2 F^{02} + \Lambda^0{}_1\Lambda^2{}_2 F^{12} = \cosh\lambda\cdot E_2 + \sinh\lambda\cdot B_3$$

$$E_3 = F^{03} \longmapsto E_3' = F'^{03} = \Lambda^0{}_\mu\Lambda^3{}_\nu F^{\mu\nu} = \Lambda^0{}_0\Lambda^3{}_3 F^{03} + \Lambda^0{}_1\Lambda^3{}_3 F^{13} = \cosh\lambda\cdot E_3 - \sinh\lambda\cdot B_2$$

$$B_1 = F^{23} \longmapsto B_1' = F'^{23} = F^{23} = B_1$$

$$B_2 = F^{31} \longmapsto B_2' = F'^{31} = \Lambda^3{}_3\left(\Lambda^1{}_0 F^{30} + \Lambda^1{}_1 F^{31}\right) = \sinh\lambda\cdot(-E_3) + \cosh\lambda\cdot B_2 = -\sinh\lambda\cdot E_3 + \cosh\lambda\cdot B_2$$

$$B_3 = F^{12} \longmapsto B_3' = F'^{12} = \Lambda^2{}_2\left(\Lambda^1{}_0 F^{02} + \Lambda^1{}_1 F^{12}\right) = \sinh\lambda\cdot E_2 + \cosh\lambda\cdot B_3$$

$\boxed{\text{Ex 2}}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$i,j,k \in \{1,2,3\}$

1. $F_{0i} = -E_i \quad , \quad F_{ij} = \varepsilon_{ijk} B_k \quad , \qquad A_0 = -A^0 = -\phi, \quad A_i = A^i \ (=(\vec{A})_i)$

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i = -\partial_i \phi - \partial_t A_i \quad \Leftrightarrow \quad \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$$

$$B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk} = \frac{1}{2}\varepsilon_{ijk}(\partial_j A_k - \partial_k A_j) \quad \Leftrightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

e.g. $B_3 = \frac{1}{2}\varepsilon_{3jk}F_{jk} = \frac{1}{2}(\varepsilon_{312}F_{12} + \varepsilon_{321}F_{21}) = \frac{1}{2}\cdot 2\varepsilon_{123}F_{12} = F_{12} = \partial_1 A_2 - \partial_2 A_1 = (\vec{\nabla}\times\vec{A})_3$.

2. The homogeneous Maxwell eqn $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ holds automatically.

$$\partial_\nu F^{\mu\nu} = J^\mu \Leftrightarrow \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\mu \Leftrightarrow \partial^\mu(\partial_\nu A^\nu) - \partial_\nu\partial^\nu A^\mu = J^\mu$$

$$\partial_\nu A^\nu = \partial_0 A^0 - \partial_i A^i = \partial_t \phi - \vec{\nabla}\cdot\vec{A} \quad , \qquad \partial_\nu \partial^\nu = -\partial_t^2 + \underbrace{\vec{\nabla}^2}_{=\Delta \text{ laplacian}}$$

$\underline{\mu=0}$: $\quad -\partial_t(\cancel{\partial_t}\phi - \vec{\nabla}\cdot\vec{A}) - (\cancel{-\partial_t^2} + \vec{\nabla}^2)\phi = \rho \quad \Leftrightarrow \quad -\vec{\nabla}^2\phi + \partial_t \vec{\nabla}\cdot\vec{A} = \rho$

$\underline{\mu=i}$: $\quad \partial_i(\partial_t\phi - \vec{\nabla}\cdot\vec{A}) - (-\partial_t^2 + \vec{\nabla}^2)A_i = j_i$

$\boxed{\text{Ex 3}}$

1. $$\frac{\partial}{\partial X_{a_1\ldots a_n}}(X^{b_1\ldots b_n}Y_{b_1\ldots b_n}) = \frac{\partial}{\partial X_{a_1\ldots a_n}}(\eta^{b_1c_1}\ldots\eta^{b_nc_n}X_{c_1\ldots c_n}Y_{b_1\ldots b_n})$$

$$= \eta^{b_1c_1}\ldots\eta^{b_nc_n}Y_{b_1\ldots b_n}\frac{\partial}{\partial X_{a_1\ldots a_n}}X_{c_1\ldots c_n} \underset{\uparrow}{=} \eta^{b_1c_1}\ldots\eta^{b_nc_n}Y_{b_1\ldots b_n}\,\delta^{a_1}_{c_1}\ldots\delta^{a_n}_{c_n} = Y^{a_1\ldots a_n}$$

$X_{c_1\ldots c_n}$ are independent

2. $$\frac{\partial}{\partial X_{a_1\ldots a_n}}(X^{b_1\ldots b_n}X_{b_1\ldots b_n}) = \eta^{b_1c_1}\ldots\eta^{b_nc_n}\frac{\partial}{\partial X_{a_1\ldots a_n}}(X_{c_1\ldots c_n}X_{b_1\ldots b_n}) =$$

$$= \eta^{b_1c_1}\ldots\eta^{b_nc_n}(\delta^{a_1}_{c_1}\ldots\delta^{a_n}_{c_n}X_{b_1\ldots b_n} + X_{c_1\ldots c_n}\delta^{a_1}_{b_1}\ldots\delta^{a_n}_{b_n}) = X^{a_1\ldots a_n} + X^{a_1\ldots a_n}$$

$$= 2X^{a_1\ldots a_n}$$

Ex 8

1. A field ϕ of charge $q \in \mathbb{Z}$ transforms as

(x-dependence left implicit)

$$\phi \longmapsto \phi' = e^{iq\alpha}\phi$$

under a $U(1)$ gauge transformation with parameter $g = e^{i\alpha} \in U(1)$, which also acts on the $U(1)$ gauge field as

$$A_\mu \longmapsto A'_\mu = A_\mu + \partial_\mu \alpha .$$

We want

$$D^{(q)}_\mu \phi \longmapsto D^{(q)\prime}_\mu \phi' = e^{iq\alpha} D^{(q)}_\mu \phi .$$

This is achieved by

$$D^{(q)}_\mu = \partial_\mu - iqA_\mu ,$$

indeed

$$\begin{aligned}(\partial_\mu - iqA_\mu)\phi \longmapsto & \left(\partial_\mu - iq(A_\mu + \partial_\mu\alpha)\right) e^{iq\alpha}\phi \\ = & \; iq\cancel{(\partial_\mu\alpha)}\, e^{iq\alpha}\phi + e^{iq\alpha}\partial_\mu\phi - iq\,(A_\mu + \cancel{\partial_\mu\alpha})\, e^{iq\alpha}\phi \\ = & \; e^{iq\alpha}\,(\partial_\mu\phi - iqA_\mu\phi) .\end{aligned}$$

2. $\bar\phi$ has charge $-q$, so

$$D^{(-q)}_\mu \bar\phi = \partial_\mu\bar\phi + iqA_\mu\bar\phi \underset{\substack{\uparrow \\ A_\mu \in \mathbb{R}}}{=} \overline{(\partial_\mu\phi - iqA_\mu)\phi} = \overline{D^{(q)}_\mu \phi} .$$