SOLUTIONS TO SELECTED EXERCISES

(not in assignments or problems classes)

Ex 1

1. Start from $F^{0i} = -F^{i0} = E_i$, $F^{ij} = \varepsilon_{ijk} B_k$ $\left(\begin{smallmatrix} F_{12} = B_3 \\ F_{23} = B_1 \\ F_{31} = B_2 \end{smallmatrix}\right)$

and lower indices with $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$. Since η is diagonal, we just pick a $-$ sign when lowering a 0 index, and a $+$ sign when lowering an $i=1,2,3$ index.

$$F_{0i} = \eta_{00}\eta_{ii} F^{0i} = -F^{0i} = -E_i \qquad \text{(no summation of indices)}$$

$$F_{ij} = \eta_{ii}\eta_{jj} F^{ij} = F_{ij} = \varepsilon_{ijk} B_k \qquad (\quad '' \quad)$$

so

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

2. Similarly, we start from $\varepsilon^{0123} = 1$ to get

$$\varepsilon_{0123} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\,\varepsilon^{0123} = -\varepsilon^{0123} = -1 .$$

The same computation applies to any other non-vanishing component of ε, since it must have exactly one time and 3 space indices:

$$\varepsilon_{\sigma(0)\sigma(1)\sigma(2)\sigma(3)} = -\varepsilon^{\sigma(0)\sigma(1)\sigma(2)\sigma(3)} = -\mathrm{sign}(\sigma) .$$

3. $\partial_\nu F^{\mu\nu} = J^\mu$

$\Updownarrow$

$\mu=0: \quad \partial_i F^{i0} = J^0 \Leftrightarrow \partial_i E_i = \rho$ $\quad (= \vec\nabla\cdot\vec E)$

$\mu=i: \quad \partial_0 F^{i0} + \partial_j F^{ij} = J^i$

$\Updownarrow$

$-\partial_t E_i + \varepsilon_{ijk}\partial_j B_k = j_i$

$\Updownarrow$

$-\partial_t \vec E + \vec\nabla\times\vec B = \vec j$

$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$

$\Updownarrow$

$\mu=0: \quad 0 = \varepsilon^{0ijk}\partial_i F_{jk} = \partial_i(\varepsilon_{ijk}F_{jk}) = 2\partial_i B_i$ $\quad \Updownarrow \quad \vec\nabla\cdot\vec B = 0$

$\mu=i: \quad 0 = \varepsilon^{i0jk}\partial_0 F_{jk} + \varepsilon^{ij0k}\partial_j F_{0k} + \varepsilon^{ijk0}\partial_j F_{k0}$

$= -\partial_0(\varepsilon_{ijk}F_{jk}) + \varepsilon_{ijk}(-\partial_j E_k - \partial_j E_k)$

$= -2\,\partial_0 B_i - 2\,\varepsilon_{ijk}\partial_j E_k$

$\Updownarrow$

$\vec 0 = \partial_t \vec B + \vec\nabla\times\vec E$

4. $F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}$

$= -2\sum_i F_{0i}^2 + 2\sum_{i<j} F_{ij}^2 = -2\sum_i E_i^2 + 2\sum_k B_k^2$

$= -2(\vec E^2 - \vec B^2)$.

5. $\rho = J^0 \mapsto \rho' = J'^0 = \Lambda^0{}_\mu J^\mu = \Lambda^0{}_0 J^0 + \Lambda^0{}_1 J^1 = \cosh\lambda\cdot\rho + \sinh\lambda\cdot j_1$

$j_1 = J^1 \mapsto j_1' = J'^1 = \Lambda^1{}_\mu J^\mu = \Lambda^1{}_0 J^0 + \Lambda^1{}_1 J^1 = \sinh\lambda\cdot\rho + \cosh\lambda\cdot j_1$

$j_{2,3} \mapsto j'_{2,3} = j_{2,3}$.

$E_1 = F^{01} \mapsto E_1' = F'^{01} = \Lambda^0{}_\mu\Lambda^1{}_\nu F^{\mu\nu} = \Lambda^0{}_0\Lambda^1{}_1 F^{01} + \Lambda^0{}_1\Lambda^1{}_0 F^{10} = (\cosh^2\lambda - \sinh^2\lambda)E_1 = E_1$

$E_2 = F^{02} \mapsto E_2' = F'^{02} = \Lambda^0{}_\mu\Lambda^2{}_\nu F^{\mu\nu} = \Lambda^0{}_0\Lambda^2{}_2 F^{02} + \Lambda^0{}_1\Lambda^2{}_2 F^{12} = \cosh\lambda\cdot E_2 + \sinh\lambda\cdot B_3$

$E_3 = F^{03} \mapsto E_3' = F'^{03} = \Lambda^0{}_\mu\Lambda^3{}_\nu F^{\mu\nu} = \Lambda^0{}_0\Lambda^3{}_3 F^{03} + \Lambda^0{}_1\Lambda^3{}_3 F^{13} = \cosh\lambda\cdot E_3 - \sinh\lambda\cdot B_2$

$B_1 = F^{23} \mapsto B_1' = F'^{23} = F^{23} = B_1$

$B_2 = F^{31} \mapsto B_2' = F'^{31} = \Lambda^3{}_3(\Lambda^1{}_0 F^{30} + \Lambda^1{}_1 F^{31}) = \sinh\lambda\cdot(-E_3) + \cosh\lambda\cdot B_2 = -\sinh\lambda\cdot E_3 + \cosh\lambda\cdot B_2$

$B_3 = F^{12} \mapsto B_3' = F'^{12} = \Lambda^2{}_2(\Lambda^1{}_0 F^{02} + \Lambda^1{}_1 F^{12}) = \sinh\lambda\cdot E_2 + \cosh\lambda\cdot B_3$

Ex 2

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$i,j,k \in \{1,2,3\}$

1. $F_{0i} = -E_i \quad , \quad F_{ij} = \varepsilon_{ijk} B_k \quad , \quad A_0 = -A^0 = -\phi, \quad A_i = A^i \; (=(\vec{A})_i)$

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i = -\partial_i \phi - \partial_t A_i \quad \Leftrightarrow \quad \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$$

$$B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk} = \frac{1}{2}\varepsilon_{ijk}(\partial_j A_k - \partial_k A_j) \quad \Leftrightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

e.g. $B_3 = \frac{1}{2}\varepsilon_{3jk} F_{jk} = \frac{1}{2}(\varepsilon_{312} F_{12} + \varepsilon_{321} F_{21}) = \frac{1}{2} \cdot 2\varepsilon_{123} F_{12} = F_{12} = \partial_1 A_2 - \partial_2 A_1 = (\vec{\nabla} \times \vec{A})_3$.

2. The homogeneous Maxwell eqn $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ holds automatically.

$$\partial_\nu F^{\mu\nu} = J^\mu \Leftrightarrow \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\mu \Leftrightarrow \partial^\mu(\partial_\nu A^\nu) - \partial_\nu \partial^\nu A^\mu = J^\mu$$

$$\partial_\nu A^\nu = \partial_0 A^0 - \partial_i A^i = \partial_t \phi - \vec{\nabla}\cdot\vec{A} \quad , \quad \partial_\nu \partial^\nu = -\partial_t^2 + \underbrace{\vec{\nabla}^2}_{=\Delta \text{ laplacian}}$$

$\underline{\mu = 0}$: $-\partial_t(\cancel{\partial_t}\phi - \vec{\nabla}\cdot\vec{A}) - (\cancel{-\partial_t^2} + \vec{\nabla}^2)\phi = \rho \quad \Leftrightarrow \quad -\vec{\nabla}^2\phi + \partial_t \vec{\nabla}\cdot\vec{A} = \rho$

$\underline{\mu = i}$: $\partial_i(\partial_t \phi - \vec{\nabla}\cdot\vec{A}) - (-\partial_t^2 + \vec{\nabla}^2) A_i = j_i$

Ex 3

1. $$\frac{\partial}{\partial X_{a_1 \dots a_n}}(X^{b_1 \dots b_n} Y_{b_1 \dots b_n}) = \frac{\partial}{\partial X_{a_1 \dots a_n}}(\eta^{b_1 c_1} \dots \eta^{b_n c_n} X_{c_1 \dots c_n} Y_{b_1 \dots b_n})$$

$$= \eta^{b_1 c_1} \dots \eta^{b_n c_n} Y_{b_1 \dots b_n} \frac{\partial}{\partial X_{a_1 \dots a_n}} X_{c_1 \dots c_n} \underset{X_{c_1 \dots c_n} \text{ are independent}}{=} \eta^{b_1 c_1} \dots \eta^{b_n c_n} Y_{b_1 \dots b_n} \delta^{a_1}_{c_1} \dots \delta^{a_n}_{c_n} = Y^{a_1 \dots a_n}$$

2. $$\frac{\partial}{\partial X_{a_1 \dots a_n}}(X^{b_1 \dots b_n} X_{b_1 \dots b_n}) = \eta^{b_1 c_1} \dots \eta^{b_n c_n} \frac{\partial}{\partial X_{a_1 \dots a_n}}(X_{c_1 \dots c_n} X_{b_1 \dots b_n}) =$$

$$= \eta^{b_1 c_1} \dots \eta^{b_n c_n}(\delta^{a_1}_{c_1} \dots \delta^{a_n}_{c_n} X_{b_1 \dots b_n} + X_{c_1 \dots c_n} \delta^{a_1}_{b_1} \dots \delta^{a_n}_{b_n}) = X^{a_1 \dots a_n} + X^{a_1 \dots a_n}$$

$$= 2 X^{a_1 \dots a_n}$$

Ex 8

1. A field ϕ of charge $q \in \mathbb{Z}$ transforms as

$$\phi \longmapsto \phi' = e^{iq\alpha}\phi$$

(x-dependence left implicit)

under a $U(1)$ gauge transformation with parameter $g = e^{i\alpha} \in U(1)$, which also acts on the $U(1)$ gauge field as

$$A_\mu \longmapsto A'_\mu = A_\mu + \partial_\mu \alpha .$$

We want

$$D^{(q)}_\mu \phi \longmapsto D^{(q)\prime}_\mu \phi' = e^{iq\alpha} D^{(q)}_\mu \phi .$$

This is achieved by

$$D^{(q)}_\mu = \partial_\mu - iqA_\mu ,$$

indeed

$$\begin{aligned}(\partial_\mu - iqA_\mu)\phi \longmapsto &\left(\partial_\mu - iq(A_\mu + \partial_\mu\alpha)\right) e^{iq\alpha}\phi \\ &= iq\cancel{(\partial_\mu\alpha)} e^{iq\alpha}\phi + e^{iq\alpha}\partial_\mu\phi - iq(A_\mu + \cancel{\partial_\mu\alpha}) e^{iq\alpha}\phi \\ &= e^{iq\alpha}\left(\partial_\mu\phi - iqA_\mu\phi\right).\end{aligned}$$

2. $\bar{\phi}$ has charge $-q$, so

$$D^{(-q)}_\mu \bar{\phi} = \partial_\mu\bar{\phi} + iqA_\mu\bar{\phi} \underset{A_\mu \in \mathbb{R}}{=} \overline{(\partial_\mu\phi - iqA_\mu)\phi} = \overline{D^{(q)}_\mu \phi} .$$

Ex 13

1. $$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_\theta = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$= -\frac{1}{2g^2}\left(-\sum_{i=1}^{3} F_{0i}^2 + \sum_{i<j} F_{ij}^2\right) + \frac{\theta}{8\pi^2}\left(-\sum_{i=1}^{3} F_{0i}\tilde{F}_{0i} + \sum_{i<j} F_{ij}\tilde{F}_{ij}\right)$$

$$= -\frac{1}{2g^2}\left(-\vec{E}^2 + \vec{B}^2\right) + \frac{\theta}{8\pi^2}\left(+\vec{E}\cdot\vec{B} + \vec{E}\cdot\vec{B}\right)$$

$$= \frac{1}{2g^2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{\theta}{4\pi^2}\vec{E}\cdot\vec{B}$$

2. $$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$\underset{\text{antisymmetry of } \varepsilon_{\mu\nu\rho\sigma}}{=} \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(2\partial_\mu A_\nu)(2\partial_\rho A_\sigma) = 2\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu\, \partial_\rho A_\sigma)$$

$$= \partial_\mu\left(2\varepsilon^{\mu\nu\rho\sigma}A_\nu \partial_\rho A_\sigma\right) = \partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right)$$

$$\underbrace{\varepsilon^{\mu\nu\rho\sigma}}_{\substack{\text{antisymmetric} \\ \text{in } \mu\rho}} A_\nu \underbrace{\partial_\mu\partial_\rho}_{\substack{\text{symmetric} \\ \text{in } \mu\rho}} A_\sigma = 0 .$$

Contribution of $\mathcal{L}_\theta$ to EoM:

$$\frac{\partial\mathcal{L}_\theta}{\partial A_\rho} - \partial_\sigma\frac{\partial\mathcal{L}_\theta}{\partial\partial_\sigma A_\rho} = -\partial_\sigma\left(\frac{\partial\mathcal{L}_\theta}{\partial F_{\mu\nu}}\frac{\partial \overbrace{F_{\mu\nu}}^{=\partial_\mu A_\nu - \partial_\nu A_\mu}}{\partial\partial_\sigma A_\rho}\right) = -\partial_\sigma\left(\frac{\theta}{8\pi^2}\tilde{F}^{\mu\nu}\left(\delta^\sigma_\mu\delta^\rho_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right)\right)$$

$$= -\frac{\theta}{4\pi^2}\partial_\sigma\tilde{F}^{\sigma\rho} = -\frac{\theta}{8\pi^2}\varepsilon^{\sigma\rho\mu\nu}\partial_\sigma F_{\mu\nu} = 0$$

by the Bianchi identity (if you don't remember it, sub in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.)

Ex 14

$$S = \int \underbrace{d^3x}_{= d^2x\, dt} \mathcal{L}\,, \qquad \mathcal{L} = -|\partial_\mu \phi|^2 - V(\phi,\bar\phi)$$

$$E = \int d^2x\, \mathcal{E}\,, \qquad \mathcal{E} = |\partial_i \phi|^2 + V(\phi,\bar\phi) \qquad \text{for static field configurations.}$$

- Euler-Lagrange eqns from stationary S:

$$0 = \frac{\partial \mathcal{L}}{\partial \bar\phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\phi} = -\frac{\partial V}{\partial \bar\phi} + \partial_\mu \partial^\mu \phi \qquad \text{(\& cplx conjugate eqn)}$$

$$= -\frac{\partial V}{\partial \bar\phi} - \partial_t^2 \phi + \partial_i \partial_i \phi$$

Restricting to static configurations:

$$0 = -\frac{\partial V}{\partial \bar\phi} + \underbrace{\partial_i \partial_i \phi}_{= \nabla^2 \phi = \Delta \phi} \qquad (1)$$

Laplacian

- Euler-Lagrange eqns from stationary E:

$$0 = \frac{\partial \mathcal{E}}{\partial \bar\phi} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \bar\phi} = \frac{\partial V}{\partial \bar\phi} - \partial_i \partial_i \phi \qquad (2)$$

So we get the same eqns (1) and (2).

Ex 16

$$\phi = \phi(\underline{x}) \qquad \nabla^2\phi - \lambda(|\phi|^2 - v^2)\phi = 0$$

$$\underline{x} = (x^1, x^2)$$

1. $x^1 + i x^2 = r e^{i\theta}$ $\qquad$ (r,θ) polar coord's on $\mathbb{R}^2$

Ansatz $\quad \phi = f(r)\, e^{i\theta}$

$$N = \frac{1}{2\pi} \oint_{S^1_\infty} \partial_i (\arg \phi)\, dx^i = \frac{1}{2\pi} \oint_{S^1_\infty} \partial_i \theta\, dx^i = \frac{1}{2\pi} \oint_{S^1_\infty} \left(\frac{\partial\theta}{\partial r} dr + \frac{\partial \theta}{\partial \theta} d\theta \right) = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1$$

where we used that the differential $d(\ldots) = \partial_i(\ldots)\, dx^i$ is independent of the choice of coordinates and that $\oint_{S^1_\infty} \frac{\partial \theta}{\partial r} dr = 0$ because $\frac{\partial\theta}{\partial r} = 0$ (and also because the integral is at fixed r).

2. Using the change of coordinates from Cartesian (x^1, x^2) to polar (r,θ) as in Ex 34 one can find (see Calculus or AMV)

$$\nabla^2 = \partial_{x^1}^2 + \partial_{x^2}^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

[In an exam, I would ask you to derive the result as a part of the question, or alternatively I'd provide the formula, depending on the length of the question.]

Next we note that $\partial_\theta^2 e^{i\theta} = -e^{i\theta}$. So

$$\nabla^2\left(f(r)\,e^{i\theta}\right) = e^{i\theta}\left(f''(r) + \frac{1}{r}f'(r) - \frac{1}{r^2}f(r)\right). \qquad (1)$$

$$\lambda\left(v^2 - |\phi|^2\right)\phi \underset{\phi = f e^{i\theta}}{\overset{\uparrow}{=}} \lambda\left(v^2 - f^2\right) f\, e^{i\theta} \qquad (2)$$

$$e^{-i\theta}\left((1)+(2)\right) = 0 \iff f'' + \frac{1}{r}f' - \frac{1}{r^2}f + \lambda\left(v^2 - f^2\right)f = 0.$$

$\boxed{\text{Ex } 19}$

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \overline{D_\mu\phi}\,D^\mu\phi - \underbrace{\frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2}_{=U(|\phi|^2)}$$

1. EoM : see Ex 10. The result is

$$0 = D_\mu D^\mu\phi - U'(|\phi|^2)\phi = D_\mu D^\mu\phi - \lambda\left(|\phi|^2 - v^2\right)\phi$$

$$\partial_\mu F^\mu{}_\nu = g^2 J_\nu = g^2\cdot i\left(\bar\phi D_\nu\phi - \overline{D_\nu\phi}\cdot\phi\right) = g^2\cdot i\left(\bar\phi\partial_\nu\phi - \overline{\partial_\nu\phi}\cdot\phi\right) + 2g^2 A_\nu|\phi|^2.$$

2. Gauge $A_0 = 0$, static solutions of the form

$$\phi(\underline{x}) = v\,e^{i\theta} f(vr)$$
$$A_j(\underline{x}) = \varepsilon_{jk}\hat{x}^k\,\frac{a(vr)}{r}.$$

$\underline{x} = (x^1, x^2) = (r\cos\theta, r\sin\theta)$

$(\hat{x}^1, \hat{x}^2) = (\cos\theta, \sin\theta)$

$$A_r = \frac{\partial x^1}{\partial r}A_1 + \frac{\partial x^2}{\partial r}A_2 = \cos\theta\,A_1 + \sin\theta\,A_2 = \cos\theta\sin\theta\,\frac{a(vr)}{r} + \sin\theta(-\cos\theta)\frac{a(vr)}{r} = 0.$$

$$A_\theta = \frac{\partial x^1}{\partial\theta}A_1 + \frac{\partial x^2}{\partial\theta}A_2 = r\left(-\sin\theta\,A_1 + \cos\theta\,A_2\right) = r\left(-\sin^2\theta - \cos^2\theta\right)\frac{a(vr)}{r} = -a(vr).$$

$$\Rightarrow F_{r\theta} = -v\,a'(vr)$$

Like in Ex 16,

$$D^2 = D_i D_i = D_r^2 + \frac{1}{r} D_r + \frac{1}{r^2} D_\theta^2 \underset{\substack{\uparrow \\ A_r = 0 \\ A_\theta = -a(vr)}}{=} \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\underbrace{\left(\partial_\theta + i a(vr)\right)^2}_{= \partial_\theta^2 + 2ia(vr)\partial_\theta - a^2(vr)}$$

acting on a field of charge 1.

Acting on $\phi = v e^{i\theta} f(vr) \equiv v e^{i\theta} f(u)$:

$$\partial_r^2 \left(v e^{i\theta} f(vr)\right) = v^3 e^{i\theta} f''(vr) = v^3 e^{i\theta} f''(u)$$

$$\frac{1}{r}\partial_r \left(v e^{i\theta} f(vr)\right) = v^2 e^{i\theta} \frac{f'(vr)}{r} = v^3 e^{i\theta}\frac{f'(u)}{u}$$

$$\frac{1}{r^2}\left(\partial_\theta + ia(vr)\right)^2 \left(v e^{i\theta} f(vr)\right) = -v e^{i\theta}\frac{f(vr)}{r^2} - 2v e^{i\theta}\frac{a(vr) f(vr)}{r^2} - v e^{i\theta}\frac{a^2(vr) f(vr)}{r^2}$$

$$= -\frac{v^3 e^{i\theta}}{u^2} f(u)\left(1 + a(u)\right)^2$$

$$-\lambda\left(|\phi|^2 - v^2\right)\phi = -\lambda v^3 e^{i\theta}\left(f^2(vr) - 1\right) f(vr) = -\lambda v^3 e^{i\theta}\left(f^2(u) - 1\right) f(u)$$

So the ϕ EoM is:

$$f''(u) + \frac{1}{u} f'(u) - \frac{1}{u^2}\left(1 + a(u)\right) f(u) + \lambda\left(1 - f^2(u)\right) f(u) = 0 . \qquad (*)$$

- The A_μ EoM is trivially satisfied $(0=0)$ for $\underline{\nu = 0}$.

For $\underline{\nu = j}$ we have $\partial_i F^i{}_j = g^2 J_j \equiv g^2 i\left(\bar\Phi \partial_j \phi - \overline{\partial_j \phi}\,\phi\right) + 2g^2 A_j |\phi|^2$
and we can either use Cartesian coordinates and then switch to polar coordinates, or use polar coordinates directly. I'll follow the latter approach. We need to remember that in polar coordinates $[g_{\mu\nu}] = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, so $[g^{\mu\nu}] = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2}\end{pmatrix}$.
Flat metric
Hence

$$F^r{}_\theta = g^{rr} F_{r\theta} = F_{r\theta} = -v\, a'(vr), \qquad F^\theta{}_r = g^{\theta\theta} F_{\theta r} = r^{-2} F_{\theta r} = v\, r^{-2} a'(vr)$$

$\boxed{j = r}$ $\quad \partial_\theta F^\theta{}_r = 0, \quad J_r \propto \mathrm{Im}\left(\bar\Phi \partial_r \phi\right) = 0 . \quad \checkmark$

$\boxed{j = \theta}$ $\quad \partial_r F^r{}_\theta = -v^2 a''(vr), \quad J_\theta = -2g^2|\phi|^2 - 2g^2 a(vr)|\phi|^2 = -2g^2\left(1 + a(vr)\right) v^2 f^2(vr)$

So the eqn is

$$a''(u) = 2g^2\left(1 + a(u)\right) f^2(u) \qquad (**)$$

The system of ODEs is $\begin{cases} (*) \\ (**) \end{cases}$