INTERLUDE :

10. KdV HIERARCHY AND CONSERVATION LAWS

[DJ § 5.2]

- **Q** Is KdV the only equation for $u(x,t)$ s.t. the operators

$$\boxed{L(u) = \frac{\partial^2}{\partial x^2} + u(x,t)} \tag{10.1}$$

at different times are isospectral?

A No, and the Lax formulation of KdV shows how to find (infinitely many) more.

- Indeed the proof of the theorem only used the equivalence

$$\underset{\text{KdV-like EQN}}{\boxed{u_t = N(u)}} \iff \underset{\text{LAX EQUATION}}{\boxed{L(u)_t = [M(u), L(u)]}} \tag{9.15}$$

IDEA : find a different $M(u)$ s.t. $[M(u), L(u)]$ is MULTIPLICATIVE, realized by multiplication by the function $N(u)$ of $u, u_x, u_{xx} \ldots$

$$u_t \longleftrightarrow L(u)_t$$
$$N(u) \longleftrightarrow [M(u), L(u)]$$

Which other conditions should $M(u)$ satisfy?

- Def : the HERMITIAN CONJUGATE (/ADJOINT) $A^\dagger$ of an operator A wrt a hermitian inner product $\langle \, , \, \rangle$ is defined by

$$\boxed{\langle \psi_1, A\psi_2 \rangle = \langle A^\dagger \psi_1, \psi_2 \rangle} \quad \forall \psi_1, \psi_2 \tag{10.2}$$

A is HERMITIAN/SELF-ADJOINT iff $A^\dagger = A$.

A is ANTIHERMITIAN iff $A^\dagger = -A$.

E.g. $L(u) = L(u)^\dagger$ is hermitian.
(in the space of square integrable functions $L^2(\mathbb{R})$)

NOTE : $[A,B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$. (10.3)

- $L = L^\dagger$ and $L_t = [M, L]$ imply

$$L_t = L_t^\dagger$$

$$\Rightarrow \quad [M, L] = [M, L]^\dagger$$

$$\underset{L^\dagger = L}{\Rightarrow} \quad [M, L] = [L, M^\dagger]$$

$$\Rightarrow \quad \boxed{[L, M + M^\dagger] = 0} \qquad (10.4)$$

Hence the hermitian part of M commutes with L and drops out of the Lax eqn.

[Note: can write a general M as

$$M = \underbrace{\tfrac{1}{2}(M + M^\dagger)}_{\text{hermitian part}} + \underbrace{\tfrac{1}{2}(M - M^\dagger)}_{\text{anti-hermitian part}}.$$
]

So wlog we can take M ANTIHERMITIAN:

$$\boxed{M(u)^\dagger = -M(u)}. \qquad (10.5)$$

- NOTE: this guarantees that $\langle \psi, \psi \rangle$ is constant under time evolution $\psi_t = M(u)\psi$.

  (* Ex: prove it.)

- So we need:

  1) $M(u)^\dagger = -M(u)$

  2) $[M(u), L(u)]$ MULTIPLICATIVE.

• For us $M(u)$ is a differential operator in x:

$$M(u) = \sum_{j=0}^{n} \alpha_j(x) D^j \,, \tag{10.6}$$

where $\alpha_j(x)$ are functions of x (and in principle t). ← dropped from notation!

However

$$\boxed{D^\dagger = -D} \tag{10.7}$$

(this is integration by parts), hence

$$(D^{2j})^\dagger = D^{2j} \,, \qquad (D^{2j-1})^\dagger = -D^{2j-1} \,. \tag{10.8}$$

We can then write the antihermitian M as

$$\boxed{M(u) = \sum_{j=1}^{m} \left(\beta_j(x) D^{2j-1} + D^{2j-1} \beta_j(x) \right)} \tag{10.9}$$

for some real functions $\beta_j(x)$. ← also functions of t, in principle.

* EXAMPLES

$\boxed{m=0}$ $\quad M(u) = 0 \Rightarrow [M(u), L(u)] = 0 = N(u)$

$$\Rightarrow \quad \boxed{u_t = 0} \,, \tag{10.10}$$

which makes $L(u) = D^2 + u$ isospectral at different t, but trivially.

E31

$\boxed{m=1}$ $\quad M(u) = \beta_1 D + D\beta_1 = \beta_{1,x} + 2\beta_1 D$

$$\begin{aligned}
[M(u), L(u)] &= [\beta_{1,x} + 2\beta_1 D, D^2 + u] \\
&= -[D^2, \beta_{1,x}] - 2[D^2, \beta_1] D + 2\beta_1 [D, u] \\
&= -\beta_{1,xxx} - 2\beta_{1,xx} D - 2\beta_{1,xx} D - 4\beta_{1,x} D^2 + 2\beta_1 u_x \\
&= -4\beta_{1,x} D^2 - 4\beta_{1,xx} D + (2\beta_1 u_x - \beta_{1,xxx})
\end{aligned} \tag{10.11}$$

This is multiplicative if the coefficients of D^2, D vanish:

D^2: $\beta_{1,x} = 0 \Rightarrow \beta_1 = \text{const.}$ (wrt x)

D^1: $\cancel{\beta_{1,xx}}^{\,0} = 0$ ✓ automatic

D^0: $2\beta_1 u_x - \cancel{\beta_{1,xxx}}^{\,0} = 2\beta_1 u_x = N(u)$

We can set $\beta_1 = \frac{1}{2}$ wlog by redefining t. The PDE is then

$$\boxed{u_t = u_x} \qquad \text{ADVECTION EQUATION} \tag{10.12}$$

which is solved by

$$u(x,t) = u(x+t, 0), \tag{10.13}$$

We could change the velocity by changing β_1.

a travelling wave moving at velocity -1. Again $L(u) = \frac{\partial^2}{\partial x^2} + u(x,t)$ at different t are isospectral, but still quite trivially: the profile of u translates rigidly at a fixed speed, and the same do the eigenfunctions.

For larger m we can always take $\beta_m = \frac{1}{2}$ (* Ex 59: show that $\beta_{m,x} = 0$).
↳ $\beta_m = \text{const}$. Then $\beta_m = \frac{1}{2}$ is an arbitrary normalization)

$\boxed{m=2}$ $\quad M(u) = D^3 + \overbrace{(\beta_1 D + D\beta_1)}^{\text{same as above}}$

$$[M(u), L(u)] = [D^3, u] - 4\beta_{1,x} D^2 - 4\beta_{1,xx} D + (2\beta_1 u_x - \beta_{1,xxx})$$

$$= (3u_x - 4\beta_{1,x}) D^2 + (3u_{xx} - 4\beta_{1,xx}) D + (u_{xxx} + 2\beta_1 u_x - \beta_{1,xxx}) \tag{10.14}$$

D^2: $\beta_{1,x} = \frac{3}{4} u_x \Rightarrow \beta_1 = \frac{3}{4} u + k$

const wrt x, but any t-dependence of k can be absorbed in a redefinition of t, x, u, so we can take $k = \text{const}$ wlog.

D^1: ✓

D^0: $u_{xxx} - \beta_{1,xxx} + 2\beta_1 u_x = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x + 2k u_x = N(u)$

$$\rightsquigarrow \boxed{u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x + 2k u_x}. \tag{10.15}$$

(or use $\beta_2 = -2$)

With $t \to -4t$ and $k=0$, this is KdV. Alternatively,

$$\boxed{\tilde{u}(x,t) \equiv u(x+8kt, -4t)} \tag{10.16}$$

satisfies KdV. (* Ex 60: check)

- This shows that KdV is the 3rd member of a hierarchy of eqns $u_t = N_m(u) = [M_m(u), L(u)]$:

$m=0$:	$u_t = 0$	$M = 0$
$m=1$:	$u_t + u_x = 0$	$M = -D$
$m=2$:	$u_t + 6uu_x + u_{xxx} = 0$	$M = -4D^3 - 3(uD + Du)$
$m=3$:	$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0$	$M = -16D^5 - 20(uD^3 + D^3u) - 5\left((3u^2 - u_{xx})D + D(3u^2 - u_{xx})\right)$
	$\vdots$	

(10.17)

This is called the KdV HIERARCHY.

10.1 – KdV hierarchy and conservation laws

- Recall chapter 4: the KdV eqn has an infinite sequence of conserved charges

$$\boxed{Q_n = \int_{-\infty}^{\infty} dx\, \rho_n} \tag{10.18}$$

where

$$\boxed{\frac{\partial}{\partial t}\rho_n + \frac{\partial}{\partial x} j_n = 0} \tag{10.19}$$

for some j_n with

$$\boxed{\left[j_n \right]_{-\infty}^{+\infty} = 0} \tag{10.20}$$

and

$$\boxed{\rho_n = u^n + \dots}\ , \tag{10.21}$$

which we obtained by means of the GARDNER TRANSFORM.

Specifically,

$$\begin{aligned} \rho_1 &= u \\ \rho_2 &= u^2 \\ \rho_3 &= u^3 - \tfrac{1}{2} u_x^2 \\ \rho_4 &= u^4 - 2u u_x^2 + \tfrac{1}{5} u_{xx}^2 \\ &\ \vdots \end{aligned} \tag{10.22}$$

- So we have 2 INFINITE SEQUENCES:

  - For the KdV eqn, an infinite sequence of conserved $Q_n[u]$;

  - Going beyond KdV, an infinite sequence of $N_n(u)$ s.t. the time evolution $u_t = N_n(u)$ leaves the eigenvalues of $L(u) = \frac{\partial^2}{\partial x^2} + u(x,t)$ constant in t.

Surprisingly, the two sequences are related.

To understand how, we need to introduce a new concept:

the FUNCTIONAL DERIVATIVE.

10.1.1 – The functional derivative

- $$\boxed{F[u] = \int_{-\infty}^{+\infty} dx \, f(u, u_x, u_{xx}, \dots)} \qquad (10.23)$$

is an example of a FUNCTIONAL of u: it maps a function $u(x,t)$ to a number (for any fixed value of t).

[Here t is a spectator and is fixed. Only x is integrated over.]

- Now consider a small variation $\delta u(x,t)$ s.t. $\delta u \xrightarrow[x\to\pm\infty]{} 0$. Then

$$F[u+\delta u] = \int_{-\infty}^{+\infty} dx \, f\left(u+\delta u, (u+\delta u)_x, (u+\delta u)_{xx}, \dots\right)$$

$$= \int_{-\infty}^{+\infty} dx \, f\left(u+\delta u, u_x + \delta u_x, u_{xx} + \delta u_{xx}, \dots\right) \qquad (10.24)$$

Taylor expand to 1st order

$$= F[u] + \int_{-\infty}^{+\infty} dx \left[\frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u_x}\delta u_x + \frac{\partial f}{\partial u_{xx}}\delta u_{xx} + \dots\right] + O((\delta u)^2)$$

Integrate by parts & $\delta u_x = (\delta u)_x$, $\delta u_{xx} = (\delta u)_{xx} \dots$

$$= F[u] + \int_{-\infty}^{+\infty} dx \, \delta u \left[\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} + \dots\right] + O((\delta u)^2)$$

and we define the FUNCTIONAL DERIVATIVE $\frac{\delta F[u]}{\delta u}$ by

$$\boxed{F[u+\delta u] - F[u] = \underbrace{\int_{-\infty}^{+\infty} dx \, \frac{\delta F[u]}{\delta u}\delta u}_{\equiv \delta F[u]} + O((\delta u)^2)} \, . \qquad (10.25)$$

Therefore

$$\boxed{\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} + \dots} \, . \qquad (10.26)$$

ASIDE: this concept is central in classical physics. One can obtain the EQUATION of MOTION by requiring that the ACTION $S[u] = \int dx\, dt\, \mathcal{L}(u, u_t, u_x, \dots)$ is stationary under all infinitesimal variations δu consistent with B.C.'s. See EXTRA READING for 4H students if you are (3H and) interested.

* **EXAMPLES :** $F[u] = \int_{-\infty}^{+\infty} dx\, f$ with

1) $f = u$: $\frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \ldots = 1$

2) $f = u^4$: $\frac{\delta F}{\delta u} = 4u^3$

3) $f = u_x^2$: $\frac{\delta F}{\delta u} = -\frac{\partial}{\partial x}(2u_x) = -2u_{xx}$.

• The CONSERVED CHARGES $Q_n[u] = \int_{-\infty}^{+\infty} dx\, \rho_n(u, u_x, u_{xx}, \ldots)$ are examples of functionals of the type (10.23).

Their functional derivatives are

$$\rho_1 = u \quad : \quad \frac{\delta Q_1}{\delta u} = 1$$

$$\rho_2 = u^2 \quad : \quad \frac{\delta Q_2}{\delta u} = 2u \qquad (10.27)$$

$$\rho_3 = u^3 - \frac{1}{2}u_x^2 \quad : \quad \frac{\delta Q_3}{\delta u} = 3u^2 + u_{xx}$$

$$\rho_4 = u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2 \quad : \quad \frac{\delta Q_4}{\delta u} = \ldots = 4u^3 + 4u_x^2 + 4uu_{xx} - 2u_x^2 + \frac{2}{5}u_{xxxx}$$

$$\vdots$$

• GENERAL CLAIM – The 2 infinite sequences are related as follows:

$$\boxed{u_t = \frac{\partial}{\partial x}\frac{\delta Q_n}{\delta u}} \iff \boxed{u_t = N_n(u)} \quad n^{\text{th}} \text{ KdV-type equation} \qquad (10.28)$$

n^{th} conserved charge in standard KdV eqn

BEWARE:
This equation is defined up to the normalization of Q_n, which can be absorbed in t. I haven't been careful in making all conventions consistent, but it can be achieved by normalizing Q_n appropriately.

Furthermore:

i) All the CHARGES $Q_n[u]$, $n=1,2,\ldots$, are CONSERVED if u evolves by *any* equation $u_t = N_m(u)$ of the KdV hierarchy.

2) One can introduce a different "time" for each equation, and package the whole hierarchy into a single $u(x, t_1, t_2, \dots)$ s.t.

$$\boxed{\frac{\partial u}{\partial t_m} = N_m(u) = \frac{\partial}{\partial x}\frac{\delta Q_m}{\delta u}} \qquad \forall\, m = 1, 2, \dots \qquad (10.29)$$

Then if we evolve for a while in t_i and then another while in t_j we get the same as if we evolve in the opposite order.

$\rightarrow$ "COMMUTING FLOWS". Very far reaching idea, but we don't have time to explore it further.

EXTRA

- For 4H students, who have seen the extra reading material:

Consider the ACTION

$$S_n[w] = \int dx\,dt \left[\tfrac{1}{2} w_x w_t - \rho_n\Big|_{u=w_x}\right] = \int dt\,dx\, \mathcal{L}(w_t, w_x, w_{xx}, \dots).$$

Requiring that it's stationary leads to the Euler-Lagrange equation

$$\begin{aligned}
0 &= -\cancel{\frac{\partial \mathcal{L}}{\partial w}} + \partial_t \frac{\partial \mathcal{L}}{\partial w_t} + \partial_x \frac{\partial \mathcal{L}}{\partial w_x} - \partial_x^2 \frac{\partial \mathcal{L}}{\partial w_{xx}} + \partial_x^3 \frac{\partial \mathcal{L}}{\partial w_{xxx}} + \dots \\
&= \partial_t\left(\tfrac{1}{2} w_x\right) + \partial_x\left(\tfrac{1}{2} w_t - \frac{\partial \rho_n}{\partial u}\Big|_{u=w_x}\right) + \partial_x^2 \frac{\partial \rho_n}{\partial u_x}\Big|_{u=w_x} - \partial_x^3 \frac{\partial \rho_n}{\partial u_{xx}}\Big|_{u=w_x} + \dots \\
&= \left(u_t - \frac{\partial}{\partial x}\frac{\delta Q_n}{\delta u}\right)\Big|_{u=w_x}
\end{aligned}$$

so we find the EQUATION of MOTION

$$u_t = \frac{\partial}{\partial x}\frac{\delta Q_n}{\delta u}$$

upon identifying $u = w_x$.