11. INVERSE SCATTERING (or "REASSEMBLY")

[DJ § 3.3, 3.4] + [Aktosun § VIII–IX]

- To conclude the INVERSE SCATTERING METHOD, we need to REASSEMBLE the KdV field $u(x,t)$, or equivalently the Schrödinger potential $V(x;t) = -u(x,t)$, from the scattering data $\{R(k), \{\mu_n, c_n\}_{n=1}^{N}\}$.

  This is STEP (C): "REASSEMBLY / INVERSE SCATTERING".

- It is surprising that this is at all possible: it's like "hearing the shape of a drum", i.e. reconstructing the shape of the drumhead from the sound harmonics it makes. Luckily this problem had already been solved by MARCHENKO (following GELFAND & LEVITAN) a few years before GGKM.

11.1 – Motivation

- Consider the wave equation for $\phi(x,z)$

$$\boxed{\phi_{xx} - \phi_{zz} = 0}. \tag{11.1}$$

We have seen many ways to solve it. One is Fourier transform. Writing

$$\begin{aligned} \phi(x,z) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \psi(x;k)\, e^{ikz} \\ \psi(x;k) &= \int_{-\infty}^{+\infty} dz\, \phi(x,z)\, e^{-ikz}, \end{aligned} \tag{11.2}$$

the wave equation (11.1) for $\phi(x,z)$ translates into

$$\boxed{\psi_{xx} + k^2 \psi = 0} \tag{11.3}$$

for its Fourier transform with respect to z, $\psi(x;k)$.

• Now suppose that we are interested in a solution of (11.3) s.t.

$$\boxed{\psi(x;k) \approx e^{-ikx} \qquad \text{as } x \to -\infty.} \tag{11.4}$$

This is obtained by choosing

$$\boxed{\phi(x,z) = \delta(x-z) + K(x,z)}\ , \tag{11.5}$$

where

$$\boxed{K(x,z)=0 \quad \text{if} \quad x<z} \tag{11.6}$$

and $K(x,z)$ is a solution of (11.1) for $\underline{z<x}$.
Indeed, if that is the case we find

$$\boxed{\psi(x;k) = e^{-ikx} + \int_{-\infty}^{x} dz\, K(x,z)\, e^{-ikz}}\ , \tag{11.7}$$

which has the correct asymptotics (11.4).

• Our aim is to solve the inverse scattering problem for the eqn

$$\boxed{\psi_{xx} + (k^2 - V)\psi = 0}\ , \tag{11.8}$$

which differs from (11.3) because of the potential $V(x)$ at time t, which we eventually identify with $-u$.
We are interested in <u>scattering</u> <u>solutions</u> (and bound state solutions), but to construct them we can use the differently normalized <u>JOST SOLUTIONS</u> $\psi_\pm(x;k)$, which have asymptotics

$$\boxed{\begin{aligned} \psi_+(x;k) &\approx e^{ikx} && \text{as } x \to +\infty \\ \psi_-(x;k) &\approx e^{-ikx} && \text{as } x \to -\infty \end{aligned}}\ . \tag{11.9}$$

For instance, we can identify the scattering solution (9.2) coming from $x = -\infty$ as

$$\psi_k(x) = T(k)\,\psi_+(x;k) = \overline{\psi_-(x;k)} + R(k)\,\psi_-(x;k)\,. \qquad (11.10)$$

complex conjugation

- Since we want to reassemble the potential $V(x)$ in (11.8) from the <u>SCATTERING DATA</u> given by the asymptotics as <u>$x\to-\infty$</u>, let's focus on the JOST SOLUTION ψ_-, which we assume to be of the form

$$\boxed{\psi_-(x;k) = e^{-ikx} + \int_{-\infty}^{x} dz\, K(x,z)\, e^{-ikz}}\,, \qquad (11.11)$$

where <u>$K(x,z)=0$ if $x<z$</u>, as in (11.6)-(11.7).

- We want to substitute the ansatz (11.11) in the Schrödinger eqn (11.8). Integrating by parts twice, using $e^{-ikz} = \frac{i}{k}\frac{\partial}{\partial z} e^{-ikz}$, and <u>assuming</u>

$$\boxed{K(x,z),\ K_z(x,z) \longrightarrow 0 \quad \text{as } z\to-\infty}\,, \qquad (11.12)$$

it is not too hard to find (∗Ex)

$$\psi_{-xx} = e^{-ikx}\left(-k^2 - ik\,K(x,x) + 2K_x(x,x) + K_z(x,x)\right) + \int_{-\infty}^{x} dz\, e^{-ikz} K_{xx}(x,z)$$

$$\psi_- = e^{-ikx}\left(1 + \frac{i}{k}K(x,x) + \frac{1}{k^2}K_z(x,x)\right) - \frac{1}{k^2}\int_{-\infty}^{x} dz\, e^{-ikz} K_{zz}(x,z)\,, \qquad (11.13)$$

where $K_{\substack{x\\z}}(x,x)$ means $K_{\substack{x\\z}}(x,z)\big|_{z=x} := \lim_{z\to x^-} K_{\substack{x\\z}}(x,z)$.

Substituting in the Schrödinger eqn (11.8) we obtain

$$\boxed{0 = e^{-ikx}\left(2\frac{d}{dx}K(x,x) - V(x)\right) + \int_{-\infty}^{x} dz\, e^{-ikz}\left(\partial_x^2 - \partial_z^2 - V(x)\right)K(x,z)}\,, \qquad (11.14)$$

which is solved if we require that

$$\boxed{K_{xx}(x,z) - K_{zz}(x,z) - V(x)\,K(x,z) = 0 \quad \underline{\forall z<x}} \qquad (11.15)$$

$$\boxed{V(x) = 2\frac{d}{dx}K(x,x) = 2\left(K_x(x,x) + K_z(x,x)\right)}\,. \qquad (11.16)$$

• REMARKS :

1) (11.15) is the Fourier transform wrt $\underline{k}$ of the Schrödinger eqn (11.8).

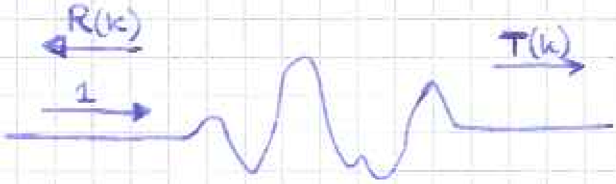
2) $V(x)$ can be obtained from K by using either (11.15) or (11.16).
It is nontrivial that the 2 equations are compatible.

3) This is still DIRECT (not INVERSE) SCATTERING.
Given the potential $V(x)$, we can solve the PDE (11.15) with the BC's (11.12) to determine $K(x,z)$ for $z<x$.
Then (11.11) gives the JOST SOLUTION $\psi_-(x;k)$.

11.2 - Recipe for inverse scattering: Marchenko equation

• We want instead to solve the inverse scattering problem:
given the SCATTERING DATA at $x=-\infty$, determine the POTENTIAL $V(x)$.
(at any fixed KdV time t)

I will not go through the derivation, which is rather long, though not too hard to follow. You can read it in § 3.3 of DJ, except that there everything is phrased in terms of scattering solutions with waves coming in from $+\infty$ and asymptotics as $x\to+\infty$, whereas we use



The upshot is the following RECIPE:

1. Construct the function

$$\boxed{F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k) e^{-ik\xi} + \sum_{n=1}^{N} c_n^2 e^{\mu_n \xi}} \qquad (11.17)$$

from the SCATTERING DATA

$$\boxed{\{ R(k), \{\mu_n, c_n\}_{n=1}^{N} \}} . \qquad (11.18)$$

$k \in \mathbb{R}$ CONTINUUM → scattering states

$k = i\mu_N$ DISCRETUUM → bound states

2. Solve the MARCHENKO EQUATION

$$\boxed{K(x,z) + F(x+z) + \int_{-\infty}^{x} dy\, K(x,y) F(y+z) = 0} \qquad (11.19)$$

to determine the unknown $K(x,z)$ $\forall z < x$.
(In addition $K(x,z) = 0$ $\forall x < z$.)

3. Finally, determine the Schrödinger potential from

$$\boxed{V(x) = 2 \frac{d}{dx} K(x,x)} . \qquad (11.20)$$

This is related to the KdV field at time t by $u = -V$.

NOTE: using incoming waves from $x = +\infty$ the relevant formulae are

$$F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k) e^{ik\xi} + \sum_{n=1}^{N} c_n^2 e^{-\mu_n \xi} ,$$

$$K(x,z) + F(x+z) + \int_{x}^{\infty} dy\, K(x,y) F(x+y) = 0 \qquad \forall z > x$$

$$V(x) = -2 \frac{d}{dx} K(x,x) = -2 \lim_{z \to x^+} \left(K_x(x,z) + K_z(x,z) \right) .$$

The final result for $V(x)$ is of course the same.

- The previous formulae are at fixed KdV time t (which is not indicated). But the time evolution is simple, using our previous results (9.36) and (9.40), which say that

$$\begin{aligned} R(k;t) &= R(k;0)\, e^{-8ik^3 t} \\ c_n(t) &= c_n(0)\, e^{-4\mu_n^3 t} \end{aligned} \qquad (11.21)$$

where the eigenvalues k^2 and $-\mu_n^2$ do not depend on time.

So to find the time evolution of u we need to apply the previous recipe starting from

$$\boxed{\begin{aligned} F(\xi;t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k;t)\, e^{-ik\xi} + \sum_{n=1}^{N} c_n^2(t)\, e^{\mu_n \xi} \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k;0)\, e^{-ik(\xi+8k^2 t)} + \sum_{n=1}^{\infty} c_n^2(0)\, e^{\mu_n(\xi-8\mu_n^2 t)} \end{aligned}} \qquad (11.22)$$

- It might be hard in practice to compute (11.22), solve (11.19) and obtain (11.20) exactly as a function of t, but the problem is solved in principle (and can for instance be solved numerically).

In particular,

- The term involving R, with the integral over k, makes it hard to compute $F(\xi;t)$ for $t \neq 0$. However, if the potential is reflectionless, the problem for $t \neq 0$ is essentially as hard as for $t = 0$. It can be solved, and it yields precisely the "pure" multi-soliton solutions that we found using Bäcklund or Hirota.

  If $R \neq 0$, it's possible to show that the term involving R goes to 0 as $t \to +\infty$ at fixed finite ξ.

- The GENERAL PICTURE, which we will confirmed by looking at examples shortly, is the following:

Ⓐ $\{\mu_n, c_n\}_{n=1}^{N} \longleftrightarrow N \overset{\text{RIGHT-MOVING}}{\text{SOLITONS}}$ "inside" initial data

Ⓑ $R(k) \longleftrightarrow$ DISPERSIVE $\overset{\text{LEFT-MOVING}}{\text{WAVES}}$ "inside initial data

The solitonic bits move to the right at velocities $8\mu_n^2$ in ξ-space. The dispersive waves move to the left at velocities $-8k^2$ in ξ-space. (The actual velocities of the corresponding components of u in real x-space are half of these. This is because F appears in the Marchenko eqn as $F(x+z)$ (and $F(x+y)$). Half goes to x and half goes to z.)

- This is a sort of "NONLINEAR FOURIER ANALYSIS".
The different components of $F(\xi)$ are superimposed linearly in (11.17). But the Marchenko function $K(x,z)$ obtained from (11.19) is not a linear superposition.

11.3 – Example 1: the single KdV soliton

- Consider a reflectionless potential, so that $R(k)=0$, with a single bound state, characterized by $\{\mu_1, c_1\} \equiv \{\mu, c\}$.
Then (at fixed t)

$$\boxed{F(\xi) = c^2 e^{\mu\xi}} \qquad (11.23)$$

and the MARCHENKO EQUATION (11.19) reads

$$\boxed{K(x,z) + c^2 e^{\mu(x+z)} + \int_{-\infty}^{x} dy\, K(x,y)\cdot c^2 e^{\mu(y+z)} = 0} \qquad (11.24)$$

which is to be solved for $z<x$. We can factor out $e^{\mu z}$ from the last two terms: (we could also factor out c^2 ...)

$$0 = K(x,z) + e^{\mu z}\underbrace{\left(c^2 e^{\mu x} + c^2\int_{-\infty}^{x} dy\, K(x,y)\, e^{\mu y}\right)}_{\text{independent of } z}, \qquad (11.25)$$

therefore

$$\boxed{K(x,z) = h(x)\, e^{\mu z}} \qquad (11.26)$$

for some function of x, $h(x)$, which satisfies (subbing (11.26) in (11.25))

$$\begin{aligned} 0 &= h(x) + c^2 e^{\mu x} + c^2 \int_{-\infty}^{x} dy\, h(x)\, e^{2\mu y} \\ &= h(x)\cdot\left[1 + c^2\int_{-\infty}^{x} dy\, e^{2\mu y}\right] + c^2 e^{\mu x} \\ &= h(x)\cdot\left[1 + \frac{c^2}{2\mu} e^{2\mu x}\right] + c^2 e^{\mu x}. \end{aligned} \qquad (11.27)$$

$$\Rightarrow \quad h(x) = -\frac{c^2 e^{\mu x}}{1 + \frac{c^2}{2\mu} e^{2\mu x}}.$$

If we let (trading c for x_0)

$$\boxed{c^2 = 2\mu\, e^{-2\mu x_0}}, \qquad (11.28)$$

we obtain

$$h(x) = -2\mu \frac{e^{\mu(x-2x_0)}}{1+e^{2\mu(x-x_0)}} \qquad (11.29)$$

and

$$K(x,z) = -2\mu \frac{e^{\mu(x+z-2x_0)}}{1+e^{2\mu(x-x_0)}}. \qquad (11.30)$$

$$e^{\mu(x+z-2x_0)} = e^{\mu[(x-x_0)+(z-x_0)]}$$

Hence

$$V(x) = 2\frac{d}{dx}K(x,x) = -4\mu \frac{d}{dx}\frac{e^{2\mu(x-x_0)}}{1+e^{2\mu(x-x_0)}}$$

$$= -4\mu \frac{d}{dx}\left(e^{-2\mu(x-x_0)}+1\right)^{-1} = -8\mu^2 \frac{e^{-2\mu(x-x_0)}}{\left(e^{-2\mu(x-x_0)}+1\right)^2}$$

$$\Longrightarrow \boxed{V(x) = -2\mu^2 \operatorname{sech}^2\left(\mu(x-x_0)\right)} \qquad (11.31)$$

Since $u=-V$, this is just a snapshot of a single KdV soliton at a time (say $t=0$) where its centre is at x_0.

(In particular, if $\mu=1$ and $c^2=2$ (see eq (9.6)), we find $V(x)=-2\operatorname{sech}^2 x$.)

- The time evolution is also easy to compute using (11.21):

$$c^2(t)=c^2(0)\,e^{-8\mu^3 t} = 2\mu e^{-2\mu x_0} e^{-8\mu^3 t} = 2\mu\, e^{-2\mu(x_0+4\mu^2 t)}, \qquad (11.32)$$

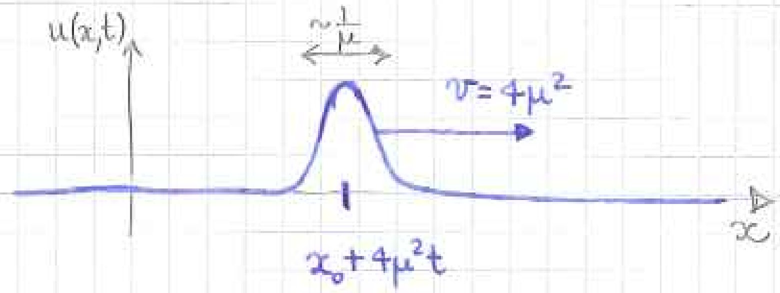
therefore as time t changes the centre of the soliton translates:

$$\boxed{x_0 \longmapsto x_0 + 4\mu^2 t}. \qquad (11.33)$$

Hence the KdV field is

$$\boxed{u(x,t) = 2\mu^2 \operatorname{sech}^2\left(\mu(x-x_0-4\mu^2 t)\right)}, \qquad (11.34)$$

in agreement with the results of the first term. (e.g. (6.3))



11.4 - Example 2: the N-soliton solution

• Now consider $R(k)=0$ again, but N bound states with $\{\mu_n, c_n\}_{n=1}^N$:

$$\boxed{F(\xi) = \sum_{n=1}^{N} c_n^2 e^{\mu_n \xi}} \tag{11.35}$$

Since $F(x+z) = \sum_{n=1}^{N} c_n^2 e^{\mu_n x} e^{\mu_n z}$ is a sum of factorized terms, we will look for a solution where $K(x,z)$ is also a sum of factorized terms. Using vector and matrix notation,

$$E(x) = \begin{pmatrix} e^{\mu_1 x} \\ \vdots \\ e^{\mu_N x} \end{pmatrix}, \quad L(x) = \begin{pmatrix} c_1^2 e^{\mu_1 x} \\ \vdots \\ c_N^2 e^{\mu_N x} \end{pmatrix}, \quad H(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_N(x) \end{pmatrix} \tag{11.36}$$

we can write

$$\boxed{F(x+z) = \overset{\text{transpose}}{E^T}(x) L(z)}, \quad = L^T(x) E(z) \tag{11.37}$$

so we look for a $K(x,z)$ of the form

$$\boxed{K(x,z) = H^T(x) L(z) = \sum_{n=1}^{N} c_n^2 h_n(x) e^{\mu_n \xi}} = L^T(z) H(x) \tag{11.38}$$

• Substituting (11.37)-(11.38) in MARCHENKO EQN, we find

$$\begin{aligned} 0 &= K(x,z) + F(x+z) + \int_{-\infty}^{x} dy\, K(x,y) F(y+z) \\ &= H^T(x) L(z) + E^T(x) L(z) + H^T(x) \int_{-\infty}^{x} dy\, L(y) E^T(y) L(z) \\ &= \left(H(x) + E(x) + \int_{-\infty}^{x} dy\, E(y) L^T(y) \cdot H(x) \right)^T L(z), \end{aligned} \tag{11.39}$$

which is clearly solved if

$$\boxed{\Gamma(x) H(x) = -E(x)}, \tag{11.40}$$

where $\Gamma(x)$ is the $N\times N$ matrix

$$\boxed{\Gamma(x) = \underbrace{\mathbb{1}}_{\text{identity matrix}} + \int_{-\infty}^{x} dy\, E(y) L^T(y)}, \tag{11.41}$$

whose matrix elements are

$$\Gamma(x)_{mn} = \delta_{mn} + \int_{-\infty}^{x} dy\, e^{\mu_m y} \cdot c_n^2 e^{\mu_n y} = \delta_{mn} + c_n^2 \frac{e^{(\mu_m+\mu_n)x}}{\mu_m+\mu_n}. \tag{11.42}$$

Note also that (11.41) implies

$$\frac{d}{dx}\Gamma(x) = E(x) L^T(x). \tag{11.43}$$

• Going back to (11.40) the vector $H(x)$ is (assuming $\det\Gamma(x)\neq 0$)

$$H(x) = -\Gamma(x)^{-1} E(x), \tag{11.44}$$

and

$$\begin{aligned} K(x,z) &= L^T(z) H(x) = -L^T(z)\Gamma(x)^{-1}E(x) \\ &= -\mathrm{tr}\left(\Gamma(x)^{-1}E(x)L^T(z)\right). \end{aligned} \tag{11.45}$$

Therefore

$$\begin{aligned} K(x,x) &= -\mathrm{tr}\left(\Gamma(x)^{-1}E(x)L^T(x)\right) \\ &\underset{(11.43)}{=} -\mathrm{tr}\left(\Gamma(x)^{-1}\frac{d}{dx}\Gamma(x)\right) \\ &= -\mathrm{tr}\left(\frac{d}{dx}\log\Gamma(x)\right) = -\frac{d}{dx}\mathrm{tr}\left(\log\Gamma(x)\right) \\ &= -\frac{d}{dx}\log\left(\det\Gamma(x)\right), \end{aligned} \tag{11.46}$$

where I used the matrix identities

$$\begin{aligned} \frac{d}{dx}\log\Gamma &= \Gamma^{-1}\frac{d}{dx}\Gamma \\ \mathrm{tr}(\log\Gamma) &= \log(\det\Gamma). \end{aligned} \tag{11.47}$$

$\boxed{Q}$ Have you seen these identities in linear algebra?

This implies that the KdV field u is

$$u = -2\frac{d}{dx}K(x,x) = 2\frac{d^2}{dx^2}\log\left(\det\Gamma(x)\right) \tag{11.48}$$

or, restoring the t-dependence hidden in Γ (through the c_n),

$$\boxed{u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\left(\det\Gamma(x;t)\right)} \tag{11.49}$$

with

$$\boxed{\Gamma(x;t)_{mn} = \delta_{mn} + c_n^2(t)\,\frac{e^{(\mu_m+\mu_n)x}}{\mu_m+\mu_n}} \tag{11.50}$$

- Eqns (11.49)-(11.50) are very similar to the N-soliton solution (6.40), which is found using Hirota's method. To see that the result is in fact the same, we can use <u>Sylvester's determinant theorem</u>, which states that

$$\det\left(\mathbb{1}+AB\right) = \det\left(\mathbb{1}+BA\right) \tag{11.51}$$

for any pair of $N\times N$ matrices A, B. For instance, taking

$$A_{mn} = e^{\mu_m x}\,\delta_{mn}\,, \qquad B_{mn} = \frac{c_n^2 e^{\mu_n x}}{\mu_m+\mu_n}\,, \tag{11.52}$$

we find

$$\begin{aligned} (AB)_{mn} &= \frac{c_n^2\, e^{(\mu_m+\mu_n)x}}{\mu_m+\mu_n} \\ (BA)_{mn} &= \frac{c_n^2\, e^{2\mu_n x}}{\mu_m+\mu_n}\,, \end{aligned} \tag{11.53}$$

therefore we can equivalently write

$$\boxed{u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\left(\det S(x;t)\right)}\,, \tag{11.54}$$

with

$$\boxed{\begin{aligned} S(x;t)_{mn} &= \frac{1}{\mu_m+\mu_n} c_n^2(t) e^{2\mu_n x} \\ &= \frac{2\mu_n}{\mu_m+\mu_n} e^{2\mu_n(x-x_{0,n}-4\mu_n^2 t)} \end{aligned}} \qquad (11.55)$$

where in the last equality I used (11.21) and

$$\boxed{c_n^2(0) = 2\mu_n e^{-2\mu_n x_{0,n}}} \,. \qquad (11.56)$$

(11.54) – (11.56) is the general form of the N-SOLITON SOLUTION.

* Ex (PROBLEMS CLASS):

Use inverse scattering to obtain the potential $V(x) = -2\beta\delta(x)$ from its scattering data (9.5), with $a = -2\beta$.

[Hint: close the integration contour with an arc at infinity in the complex plane to compute $\int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k) e^{-ik\xi}$.

Discuss the cases $\beta < 0$ and $\beta > 0$ separately.]

DON'T TRY TO EVOLVE THE SYSTEM IN TIME. IT GETS TOO HARD!