12. INTEGRABLE SYSTEMS IN CLASSICAL MECHANICS

- So far we have (secretly) looked at integrable infinite-dimensional classical systems: classical field theories in 1 SPACE + 1 TIME dimensions, which can be thought of as the continuum limit of a lattice of infinitely many coupled oscillators (see section 2.3), with infinitely many conservation laws.

- Many of the methods that we have developed in this course, in particular the LAX PAIR method, also apply to finite-dimensional classical integrable Hamiltonian system.

- A FINITE-DIMENSIONAL HAMILTONIAN SYSTEM is defined by:

  - A set of (generalized) coordinates $q_{i=1,\dots,n}(t)$ and momenta $p_{i=1,\dots,n}(t)$ which specify the configuration of the system at time t;
    (The space parametrized by the "canonical coordinates" q, p is called "PHASE SPACE")
  - A function $H(q,p)$, called the Hamiltonian;
  - Time evolution equations: $\left(\dot{} \equiv \frac{d}{dt} \right)$

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}} \qquad \text{HAMILTON'S EQUATIONS} \qquad (12.1)$$

* EXAMPLE:

n particles moving in 1 dimension under conservative forces

$$\boxed{H(q,p) = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q_1, \dots, q_n)} \qquad (12.2)$$

mass of the i^{th} particle

Hamilton's eqns are:

$$\boxed{\dot{q}_i = \frac{p_i}{m_i} \quad , \qquad \dot{p}_i = -\frac{\partial V(q)}{\partial q_i}} \quad , \tag{12.3}$$

which are equivalent to the 2^{nd} order Newton's eqns

$$\boxed{m_i \ddot{q}_i = -\underbrace{\frac{\partial V(q)}{\partial q_i}}_{\text{Force on } i^{th} \text{ particle}}} \quad . \tag{12.4}$$

• One can associate to a Hamiltonian system a POISSON BRACKET $\{ , \}$, a bilinear antisymmetric form on the space of functions of q, p:

$$\boxed{\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)} = -\{g, f\} \tag{12.5}$$

• Then Hamilton's eqns (12.1) imply that a function $f(q,p)$ which does not depend explicitly on t (it only depends on t implicitly through $q(t)$ and $p(t)$) evolves as

$$\frac{d}{dt} f(q(t), p(t)) = \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \right)$$

$$\underset{(12.1)}{=} \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) ,$$

that is

$$\boxed{\frac{d}{dt} f(q,p) = \{H(q,p), f(q,p)\}} \quad . \tag{12.6}$$

(If f depends on t explicitly, namely $f = f(q(t), p(t); t)$, then

$$\frac{d}{dt} f = \frac{\partial}{\partial t} f + \{H, f\} \, .$$
)

- Functions $F(q,p)$ which do not depend explicitly on time and have vanishing Poisson bracket with the Hamiltonian are <u>CONSERVED</u>:

$$\boxed{\{F(q,p), H(q,p)\} = 0 \quad \Rightarrow \quad \frac{d}{dt} F(q(t), p(t)) \underset{(12.6)}{=} 0} \tag{12.7}$$

In particular the Hamiltonian is always conserved, by antisymmetry of $\{\,,\,\}$:

$$\frac{d}{dt} H = \{H, H\} \underset{(12.5)}{=} 0 . \tag{12.8}$$

Hence

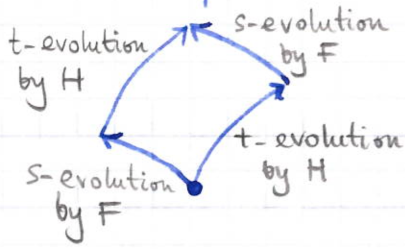
$$\boxed{H(q(t), p(t)) = \underbrace{E}_{\text{ENERGY}} = \text{const.}} \qquad \text{CONSERVATION OF ENERGY} \tag{12.9}$$

- **<u>NOTE</u>**:

If $\{F, H\} = 0$, then not only $F(q,p)$ is conserved under the time evolution (12.1) with Hamiltonian $H(q,p)$, but also $H(q,p)$ is conserved under a different time evolution with the different time s and Hamiltonian $F(q,p)$:

$$\begin{cases} \dfrac{d}{ds} q_i = \dfrac{\partial F}{\partial p_i} \\[2ex] \dfrac{d}{ds} p_i = -\dfrac{\partial F}{\partial q_i} \end{cases} \quad \Rightarrow \quad \frac{d}{ds} H(q,p) = \{F(q,p), H(q,p)\} = 0 . \tag{12.10}$$

That means that we can evolve along "2 times", and evolving first along t and then along s or first along s and then along t we end up at the same point:



In fancy language, F and H s.t. $\{F,H\}$ are said to be "IN INVOLUTION" and they generate "COMMUTING FLOWS", where one flow is the t-evolution with Hamiltonian H and the other flow is the s-evolution with Hamiltonian F.

- **Def.** — A HAMILTONIAN SYSTEM $q_{i=1,\dots,n}$, $p_{i=1,\dots,n}$, $H(q_i,p_i)$ is called COMPLETELY INTEGRABLE if it has n INDEPENDENT CONSERVED QUANTITIES $Q_i(q,p)$, satisfying $\{Q_i,H\}=0$, which are mutually in involution, that is

$$\boxed{\{Q_i,Q_j\}=0 \quad \forall\, i,j=1,\dots,n} \,. \tag{12.11}$$

(One of these conserved quantities is always the Hamiltonian.)

This means that it is possible to find a set of coordinates φ_i and momenta Q_i such that the Hamiltonian only depends on Q and not on φ:

$$H=H(Q) \quad \Rightarrow \quad \begin{cases} \dot{\varphi}_i = \dfrac{\partial H}{\partial Q_i} \\ \dot{Q}_i = -\dfrac{\partial H}{\partial \varphi_i} = 0 \end{cases} \tag{12.12}$$

These are called "ACTION-ANGLE VARIABLES" (φ_i: angle variables, Q_i: action variables). (The name is because if the surfaces of constant H are compact, then φ_i parameterize periodic orbits and therefore can be thought of as angular variables.)

- The n CONSERVED QUANTITIES Q_i are the finite-dimensional analogues of the ∞-ly many conserved charges of the KdV hierarchy.

• What is interesting for us is that the INTEGRABILITY of a classical system can be established by constructing a LAX PAIR L, M satisfying

$$\boxed{\dot{L} = [M, L]} \qquad \text{LAX EQUATION} \tag{12.13}$$

This is like for KdV, but now L and M are $n \times n$ matrices instead of differential operators (which can be thought of as ∞-dimensional matrices in an appropriate basis).

We will see that the n conserved quantities are the eigenvalues $\lambda_{i=1,\dots,n}$ of the Lax matrix L.

(To see that they are in involution requires more work. We won't see that.)

• The Lax equation (12.13), where L and M are functions of time t, can be solved formally by

$$\boxed{L(t) = U(t)\, L(0)\, U(t)^{-1}} \,, \tag{12.14}$$

where the time evolution operator $U(t)$ is the unique solution of

unitary ($U^{+} = U^{-1}$) if $M = -M^{+}$.

$$\boxed{\begin{cases} \dot{U}(t) = M(t)\, U(t) \\ U(0) = \mathbb{1} \,. \end{cases}} \tag{12.15}$$

$n \times n$ identity matrix

Indeed

$$\dot{L} \underset{(12.14)}{=} \frac{d}{dt}\left(U L(0) U^{-1}\right) = \dot{U} L(0) U^{-1} - U L(0) U^{-1} \dot{U} U^{-1}$$

$$\frac{d}{dt} U^{-1} = U^{-1} \frac{dU}{dt} U^{-1}$$

$$= \underbrace{\dot{U} U^{-1}}\, \underbrace{U L(0) U^{-1}} - \underbrace{U L(0) U^{-1}}\, \underbrace{\dot{U} U^{-1}} \tag{12.16}$$

$$\underset{\substack{(12.14)\\(12.15)}}{=} ML - LM \,.$$

- The formal solution (12.14) implies that the eigenvalues of the Lax matrix do not depend on time, as in the ∞-dimensional case of KdV-like equations. To see that, consider

$$\boxed{P_L(\lambda) = \det(\lambda \mathbb{1} - L).} \qquad \underline{\text{CHARACTERISTIC}} \quad \underline{\text{POLYNOMIAL OF } L} \qquad (12.17)$$

Recall from Linear Algebra:

This is a degree n monic polynomial ("monic": $P_L(\lambda) = \lambda^n + O(\lambda^{n-1})$) whose roots are the n eigenvalues $\lambda_{i=1,\dots,n}$ of L.

Indeed L is a hermitian (and often real) matrix, which can be diagonalized by a unitary transformation:

$$L = V \Lambda V^{-1}, \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}. \qquad (12.18).$$

Then

$$\begin{aligned} P_L(\lambda) &= \det(\lambda \mathbb{1} - V \Lambda V^{-1}) = \\ &= \det(\lambda V V^{-1} - V \Lambda V^{-1}) \\ &= \det\left(V (\lambda \mathbb{1} - \Lambda) V^{-1}\right) \qquad (12.19) \\ &= \det(V) \det(\lambda \mathbb{1} - \Lambda) \det(V^{-1}) \\ &= \det(\lambda \mathbb{1} - \Lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) \equiv P_\Lambda(\lambda) = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} \dots + \underbrace{(-1)^n \prod_i \lambda_i}_{c_n} \end{aligned}$$

Since the time evolution is similarly given by conjugation, the same argument shows that

$$\begin{aligned} P_{L(t)}(\lambda) &= \det(\lambda \mathbb{1} - U(t) L(0) U^{-1}) \\ &= \det(\lambda \mathbb{1} - L(0)) = P_{L(0)}(\lambda), \end{aligned} \qquad (12.20)$$

which implies that the eigenvalues λ_i of $L(t)$ are independent of time.

Equivalently, we can think of the n conserved quantities as

$$\boxed{c_k = \sum_{1\le i_1 < i_2 < \dots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} \quad , \quad k=1,\dots,n} \quad , \tag{12.21}$$

the coefficients of the characteristic polynomial, or as

$$\boxed{s_k = \sum_{i=1}^{n} \lambda_i^k = \mathrm{tr}(L^k) \quad , \quad k=1,\dots,n \, .} \tag{12.22}$$

(These are related by Newton identities

$$k\, c_k = \sum_{i=1}^{k} (-1)^{i-1} c_{k-i}\, s_i \, .$$
)

- Finally, note that the eigenvalue equation for L, namely

$$\boxed{L(t)\,\psi(t) = \lambda\,\psi(t)} \tag{12.23}$$

is solved formally by

$$\boxed{\psi(t) = U(t)\,\psi(0)} \, , \tag{12.24}$$

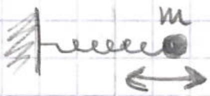
where $\psi(0)$ is the eigenfunction at $t=0$. Indeed

$$\begin{aligned} L(t)\,\psi(t) &\underset{\substack{(12.14)\\(12.24)}}{=} U(t)L(0)U(t)^{-1}U(t)\psi(0) \\ &= U(t)\,L(0)\,\psi(0) = \lambda\,U(t)\psi(0) = \lambda\,\psi(t) \, . \end{aligned} \tag{12.25}$$

12.1 – Lax pair for the simple harmonic oscillator

- The Hamiltonian for a simple harmonic oscillator $(n=1)$ is

$$\boxed{H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2}\,, \tag{12.26}$$



which leads to Hamilton's eqns

$$\boxed{\dot{q} = \frac{p}{m}\,, \qquad \dot{p} = -m\omega^2 q}\,. \tag{12.27}$$

- Hamilton's eqns (12.27) are equivalent to the Lax eqn (12.13) with

$$L = \begin{pmatrix} p & m\omega q \\ m\omega q & -p \end{pmatrix}, \qquad M = \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12.28}$$

Indeed (* Ex : do the computations)

$$\dot{L} = \begin{pmatrix} \dot{p} & m\omega\dot{q} \\ m\omega\dot{q} & -\dot{p} \end{pmatrix} = [M, L] = \begin{pmatrix} -m\omega^2 q & \omega p \\ \omega p & m\omega^2 q \end{pmatrix} \tag{12.29}$$

is nothing but (12.27).

- Since in this case M is t-independent, the time evolution operator $U(t)$ defined by (12.15) is simply

$$\boxed{U(t) = e^{Mt}}\,, \tag{12.30}$$

where the exponential of the matrix Mt is defined by Taylor expansion:

$$e^{Mt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n. \tag{12.31}$$

By the Cayley-Hamilton theorem, this should be expressible as a polynomial of degree $2-1$ in M. This is found by noticing that

$$M^2 = -\left(\frac{\omega}{2}\right)^2 \mathbb{1} \Rightarrow M^{2k} = (-1)^k \frac{\omega^{2k}}{2^{2k}} \mathbb{1}, \quad M^{2k+1} = (-1)^k \frac{\omega^{2k}}{2^{2k}} M, \qquad (12.32)$$

$$= (-1)^k \frac{\omega^{2k+1}}{2^{2k+1}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

hence

$$U(t) = e^{Mt} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} M^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} M^{2k+1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\omega t}{2}\right)^{2k} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\omega t}{2}\right)^{2k+1}$$

$$\underset{(*)}{=} \cos\left(\frac{\omega t}{2}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\left(\frac{\omega t}{2}\right) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (12.33)$$

$$= \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & -\sin\left(\frac{\omega t}{2}\right) \\ \sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix}, \qquad \leftarrow \text{rotation by } \frac{\omega t}{2}$$

where I used the Taylor series for $\sin x$ and $\cos x$ in $(*)$.

- This means that

$$L(t) = \begin{pmatrix} p(t) & m\omega q(t) \\ m\omega q(t) & -p(t) \end{pmatrix} = U(t) L(0) U(t)^{-1}$$

$$= \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & -\sin\left(\frac{\omega t}{2}\right) \\ \sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} \begin{pmatrix} p(0) & m\omega q(0) \\ m\omega q(0) & -p(0) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & \sin\left(\frac{\omega t}{2}\right) \\ -\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} \qquad (12.34)$$

$$= \dots = \begin{pmatrix} p(0)\cos(\omega t) - m\omega q(0)\sin(\omega t) & p(0)\sin(\omega t) + m\omega q(0)\cos(\omega t) \\ p(0)\sin(\omega t) + m\omega q(0)\cos(\omega t) & -p(0)\cos(\omega t) + m\omega q(0)\sin(\omega t) \end{pmatrix},$$

which means that

$$\begin{cases} q(t) = q(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t) \\ p(t) = p(0)\cos(\omega t) - m\omega q(0)\sin(\omega t) \end{cases} \qquad (12.35)$$

- In this case $n=1$, so there is a single conserved quantity, which should just be the Hamiltonian. Indeed $\mathrm{Tr}(L)=0$ and

$$\mathrm{Tr}(L^2) = \mathrm{Tr}\begin{pmatrix} p^2+m^2\omega^2q^2 & 0 \\ 0 & p^2+m^2\omega^2q^2 \end{pmatrix}$$

$$= 2\left(p^2+m^2\omega^2q^2\right) = 4m\cdot H(q,p) \,. \qquad (12.35)$$

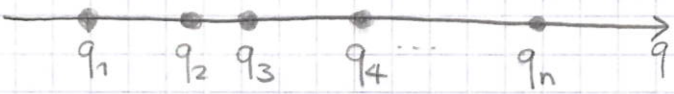
is the only independent conserved quantity.

12.2 – Lax pair for the Toda lattice

- The previous example was a bit trivial. A less trivial integrable system is the FINITE TODA LATTICE, which describes n particles on a line interacting with their nearest neighbours only.
Let's take the particles to have equal masses $m_i = 1$ for simplicity.
The Hamiltonian is

$$\boxed{H(q,p) = \sum_{i=1}^{n}\left(\frac{p_i^2}{2} + e^{-(q_i-q_{i-1})}\right)} \qquad (12.36)$$

where $\quad q_0 \equiv -\infty < q_1 < q_2 < \dots < q_n < q_{n+1} \equiv +\infty \,.$



Hamilton's eqns are

$$\boxed{\begin{aligned} \dot{q}_i &= p_i \\ \dot{p}_i &= e^{-(q_i-q_{i-1})} - e^{-(q_{i+1}-q_i)} \end{aligned}} \,, \qquad (12.37)$$

a system of coupled differential equations.

• The Lax pair is most simply formulated in terms of FLASCHKA's variables

$$\boxed{a_i = \frac{1}{2} e^{-(q_{i+1}-q_i)/2} \;,\quad b_i = -\frac{1}{2} p_i} \;, \tag{12.38}$$

which satisfy the Hamilton's eqns

$$\begin{aligned} \dot{a}_i &= -\frac{1}{4} e^{-(q_{i+1}-q_i)/2} (p_{i+1}-p_i) = a_i(b_{i+1}-b_i) \\ \dot{b}_i &= -\frac{1}{2}\left(e^{-(q_i-q_{i-1})} - e^{-(q_{i+1}-q_i)}\right) = 2\left(a_i^2 - a_{i-1}^2\right). \end{aligned} \tag{12.39}$$

Then the Lax pair is

$$L = \begin{pmatrix} b_1 & a_1 & 0 & & & \\ a_1 & b_2 & a_2 & & 0 & \\ & a_2 & b_3 & a_3 & & \\ & 0 & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_n & a_n \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & a_2 & & 0 & \\ & -a_2 & 0 & a_3 & & \\ & 0 & \ddots & \ddots & \ddots & \\ & & & -a_{n-1} & 0 & a_n \end{pmatrix} \tag{12.40}$$

* Ex : check that $\dot{L} = [M, L] \Leftrightarrow (12.39)$.

• There are n conserved quantities

$$\boxed{Q_k = \mathrm{Tr}(L^k) \;,\quad k = 1, \dots, n} \;, \tag{12.41}$$

the first few of which are (* Ex: compute Q_2)

$$Q_1 = \mathrm{Tr}(L) = \sum_{i=1}^{n} b_i = -\frac{1}{2}\sum_{i=1}^{n} p_i$$ = Total momentum

$$Q_2 = \mathrm{Tr}(L^2) = \dots = \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n-1} a_i^2$$

$$= \frac{1}{2}\boxed{\left(\sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{-(q_{i+1}-q_i)}\right)} \tag{12.42}$$

= Hamiltonian (or total energy)

$$Q_3 = \mathrm{Tr}(L^3) = \dots = \sum_{i=1}^{n} b_i^3 + 3\sum_{i=1}^{n-1} a_i^2 (b_i + b_{i+1})$$

$$= \frac{1}{8}\left(\sum_{i=1}^{n} p_i^3 - 3\sum_{i=1}^{n-1} e^{-(q_{i+1}-q_i)}(p_i + p_{i+1})\right)$$

$$\vdots$$

- Interestingly, in the limit $n \to \infty$ one gets the INFINITE TODA LATTICE, which describes an infinite number of particles on a line. This system has SOLITONS, which can be derived by a number of methods, including the inverse scattering methods.

Note that here the analogy is

$$i \in \mathbb{Z} \quad \longleftrightarrow \quad x \in \mathbb{R}$$

$$q_i \in \mathbb{R} \quad \longleftrightarrow \quad u(x) \in \mathbb{R} \qquad \text{(at fixed } t\text{)}$$

so space is discretized, whereas time is continuous.

It is not hard to check that the following is a solution:

$$\boxed{q_\ell(t) = q_0 - \ln \frac{1+\gamma e^{-2k\ell \pm 2\sinh(k)\cdot t}}{1+\gamma e^{-2k(\ell-1)\pm 2\sinh(k)\cdot t}}} \qquad \text{1-SOLITON SOLUTION} \tag{12.43}$$

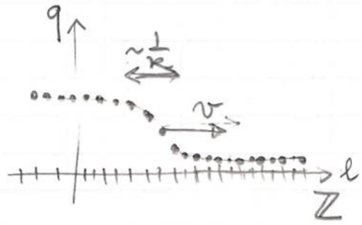
I use ℓ instead of i to avoid confusion with the imaginary unit $\sqrt{-1}$

$$(\gamma, k > 0)$$

parametrized by κ

It describes a soliton moving through $\mathbb{Z}$ with

$$\boxed{\begin{aligned} \text{velocity} &= \pm \frac{\sinh(\kappa)}{\kappa} \\ \text{width} &\sim \frac{1}{\kappa} \end{aligned}} \tag{12.44}$$



(Note: the faster the soliton, the narrower it is.)

One can similarly construct the general <u>N-SOLITON SOLUTION</u>

$$\boxed{q_\ell(t) = q_0 - \ln \frac{\det\left(\mathbb{1}_{N\times N} + C_\ell(t)\right)}{\det\left(\mathbb{1}_{N\times N} + C_{\ell-1}(t)\right)}} \tag{12.45}$$

where $\{C_\ell(t)\}_{\ell\in\mathbb{Z}}$ is a family of $N\times N$ matrices, depending on the "spatial" coordinate ℓ and the temporal coordinate t, given by

$$\boxed{\underbrace{\left(C_\ell(t)\right)_{ij}}_{1\le i,j\le N} = \frac{\sqrt{\gamma_i\gamma_j}}{1-e^{-(\kappa_i+\kappa_j)}}\, e^{-(\kappa_i+\kappa_j)\ell - (\sigma_i \sinh(\kappa_i) + \sigma_j \sinh(\kappa_j))t}} \tag{12.46}$$

with $\kappa_i, \gamma_i > 0$ and $\sigma_i = \pm 1$.