7. OVERVIEW OF THE INVERSE SCATTERING METHOD

[DJ §4.2 ; A pp.1-4]　　　　A ≡ Aktosun's review (link in DUO)

7.1 - Initial Value Problems

- SO FAR: a variety of methods to construct particular solutions.

- QUESTION: Is it possible to find a general solution?

  More precisely:

IVP: Given a wave equation and sufficient initial data at $t=0$ $(u(x,0), u_t(x,0), \dots)$, find $u(x,t)$ at all later times $t>0$.

This is an "INITIAL VALUE PROBLEM".

In order for the solution to be unique, we need sufficient initial data.

- Eqn 1st order in t (e.g. KdV): must specify $u(x,0)$.
- Eqn 2nd order in t (e.g. s-G): " " $u(x,0)$, $u_t(x,0)$.

⋮

[Why? Because we can use the equation to determine higher t-derivatives. E.g. in KdV given $u(x,0)$ we can determine $u_t(x,0)$. ← Not an independent piece of (initial) data.]

- But Given sufficient initial data, can we actually construct $u(x,t)$?

So far we don't know how to do it. If the initial data happens to be that of one of the particular solutions we constructed earlier, then YES by uniqueness, otherwise NO.

- E.g. in KdV, what if

  (a) $u(x,0) = 2 \cdot \operatorname{sech}^2 x$

  (b) $u(x,0) = 2.001 \cdot \operatorname{sech}^2 x$

  (c) $u(x,0) = 6 \cdot \operatorname{sech}^2 x$?

\- Case (a) is a snapshot of a 1-soliton solution at $t=0$.

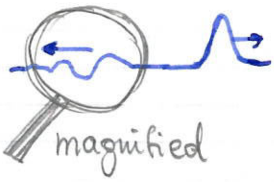
$\rightarrow$ $u(x,t) = 2 \operatorname{sech}^2(x-4t)$ (by uniqueness)

1-soliton moving to the right

\- Case (b) has very similar initial data, but evolves into

$\rightarrow$ 1 soliton + dispersive waves.

moving to the right ... moving to the left



\- Case (c) turns out to be the $t=0$ snapshot of a 2-soliton sol'n:

$\rightarrow$ 2 solitons.

moving to the right

and we can write down $u(x,t)$ by uniqueness, if we recognize that it's the snapshot of a 2-soliton solution. But what if we don't?

- Inverse Scattering will allow some understanding of situations like (b), and a much more complete idea of when things like (c) can happen. (All analytically!)

- To get an idea of how it might work, let's first look at linear wave equations.

7.2 – Linear Initial Value Problems

- For linear wave eqns, the general solution is a linear transformation of the initial data.

✱ EXAMPLE 1: Heat equation

$$\boxed{u_t = u_{xx}} \qquad (x \in \mathbb{R},\ t>0) \qquad (7.1)$$

Given $u(x,0)$, we can write the general solution at $t>0$ as

$$\boxed{u(x,t) = \int_{-\infty}^{+\infty} dx' \, \frac{1}{\sqrt{4\pi t}} \, e^{-\frac{(x-x')^2}{4t}} \, u(x',0)}\ , \qquad (7.2)$$

which is a linear transform.

(This is actually a Green's function)

(✱ EX : check that (7.2) solves (7.1) and reduces to $u(x,0)$ at $t=0$.)

✱ EXAMPLE 2: Klein-Gordon equation

$$\boxed{u_{tt} - u_{xx} + m^2 u = 0} \qquad (x \in \mathbb{R},\ t>0) \qquad (7.3)$$

2^{nd} order in t, so we must specify the initial data

$$\begin{aligned} u(x,0) &= \alpha(x) \\ u_t(x,0) &= \beta(x)\ . \end{aligned} \qquad (7.4)$$

- We can solve by Fourier transform wrt x (see handout):

$$\boxed{u(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \, \tilde{u}(k,t)\, e^{ikx}}\ . \qquad (7.5)$$

$$\tilde{u}(k,t) = \int_{-\infty}^{+\infty} dx \, u(x,t)\, e^{-ikx}$$

Then the KG eqn becomes the simpler ODE (in t)

$$\boxed{\tilde{u}_{tt} + (k^2+m^2)\,\tilde{u} = 0} \qquad (7.6)$$

for the FT $\tilde{u}(k,t)$.

Q: Is it clear why?

- The general solution of (7.6) is

$$\tilde{u}(k,t) = A(k)\,e^{i\omega t} + B(k)\,e^{-i\omega t},$$

where $\omega = \omega(k) \equiv \sqrt{k^2+m^2}$. (dispersion relation) (7.7)

Imposing initial conditions at $t=0$,

$$\begin{aligned} \tilde{\alpha}(k) &\equiv \tilde{u}(k,0) = A(k) + B(k) \\ \tilde{\beta}(k) &\equiv \tilde{u}_t(k,0) = i\omega\,(A(k)-B(k)), \end{aligned} \qquad (7.8)$$

So

$$\boxed{\begin{aligned} \tilde{u}(k,t) &= \frac{1}{2}\left(\tilde{\alpha}(k) + \frac{\tilde{\beta}(k)}{i\omega}\right)e^{i\omega t} + \frac{1}{2}\left(\tilde{\alpha}(k) - \frac{\tilde{\beta}(k)}{i\omega}\right)e^{-i\omega t} \\ &= \tilde{\alpha}(k)\cos(\omega t) + \frac{1}{\omega}\tilde{\beta}(k)\sin(\omega t) \end{aligned}} \qquad (7.9)$$

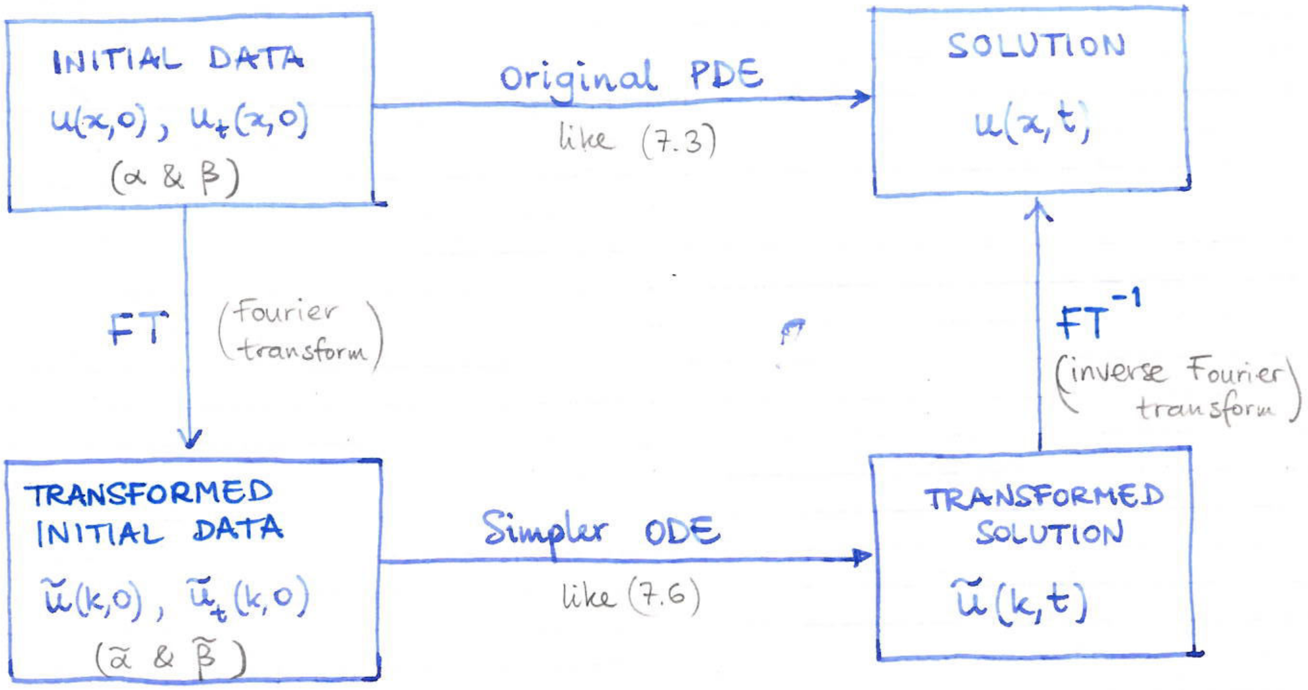
- Doing an inverse Fourier transform,

$$\boxed{\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\left[\tilde{\alpha}(k)\cos(\omega t) + \frac{1}{\omega}\tilde{\beta}(k)\sin(\omega t)\right]e^{ikx} \\ &= \iint_{-\infty}^{+\infty} \frac{dk\,dx'}{2\pi}\left[\underline{u(x',0)}\cos(\omega t) + \frac{1}{\omega}\underline{u_t(x',0)}\sin(\omega t)\right]e^{ik(x-x')} \end{aligned}} \qquad (7.10)$$

where $\omega = \omega(k) = \sqrt{k^2+m^2}$. Again, linear in $u(x,0)$ and $u_t(x,0)$.

- KEY POINT: the Fourier transformed data $\tilde{u}(k,t)$ evolved separately (for each k) and in a simple way under the transformed equation (7.6).

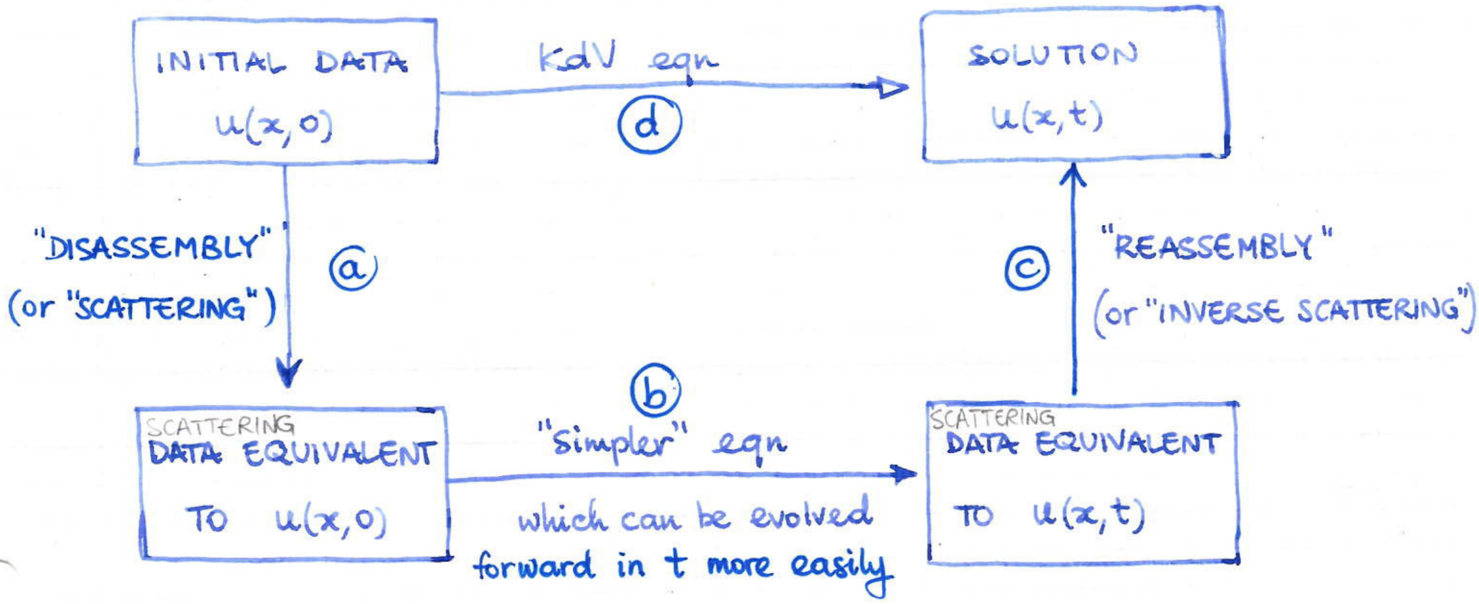
- **GENERAL PICTURE:**



and we follow $\downarrow \rightarrow \uparrow$ instead of the harder $\longrightarrow$ to find the solution at later times $u(x,t)$.

7.3 - Outline of the method to solve the IVP for KdV

- We follow a similar logic, but all the steps are non-trivial now:



Again we follow the roundabout $\downarrow$ a $\rightarrow$ b $\rightarrow$ c $\uparrow$ instead of the direct/harder $\overrightarrow{d}$.

7.3.1 – The KdV-Schrödinger connection

- We will follow the route taken by Gardner, Greene, Kruskal & Miura, who discovered the method in the late 60's.
  We want solve the KdV eqn on the full line:

$$\boxed{u_t + 6uu_x + u_{xxx} = 0} \qquad (x \in \mathbb{R},\ t > 0) \tag{7.11}$$

with the initial condition

$$\boxed{u(x,0) = f(x)} \ , \tag{7.12}$$

where $f(x)$ is sufficiently localised in space.

- Recall the (generalised) Miura transform introduced in chapter 4. (in preparation for the Gardner transform)
  If $v(x,t)$ satisfies

$$v_t + 6(\lambda - v^2)v_x + v_{xxx} = 0 \ , \tag{7.13}$$

then

$$\boxed{u = \lambda - v^2 - v_x} \tag{7.14}$$

satisfies the KdV equation (7.11).

- Now we think about this backwards: take u to be known, and try to solve (7.14) for v. (We can ignore (7.13) from now on.)

(7.14) is a Riccati equation (1st order ODE quadratic in the unknown), and can be rewritten as a linear 2nd order ODE in a standard way.
Write

$$\boxed{v = \frac{\psi_x}{\psi}} \ . \tag{7.15}$$

Then (7.14) becomes

$$u = \lambda - \cancel{\frac{\psi_x^2}{\psi^2}} - \frac{\psi_{xx}}{\psi} + \cancel{\frac{\psi_x^2}{\psi^2}}$$

or

$$\boxed{\psi_{xx} + u\psi = \lambda\psi} \, . \qquad (7.16)$$

- This equation is remarkable (and attracted the attention of GGKM) because it is nothing but the time-independent Schrödinger eqn, the quantum-mechanical equation for a particle in a potential $-u$.

The QM interpretation is not too important. What matters is that a lot was known about equations of this type ("Schrödinger problems").

(also related to well-known "Sturm-Liouville problems")

7.3.2 - Recipe for constructing the KdV solution (in 3 steps)

ⓐ DISASSEMBLY (SCATTERING)

The differential eqn is in x.

Note : t only appears as a parameter, through $u(x,t)$, in (7.16).

Start at $t=0$: initial data $u(x,0)$ plays the role of a potential.

For each eigenvalue λ, ψ is a different eigenfunction (like "eigenvector", but it's a function). It describes the scattering of a particle off the potential with certain reflection and transmission coefficients (i.e. the asymptotic values of ψ at $x \to \pm\infty$). The set of these coefficients, for different values of λ, are the INITIAL SCATTERING DATA $S(0)$.

• ANALOGY WITH LINEAR PROBLEMS AND FOURIER TRANSFORM :

	KdV		Linear PDE	
	λ	$\longleftrightarrow$	k	
Scattering data	$S(0)$	$\longleftrightarrow$	$\{\tilde{u}(k,0)\}$ or $\{\tilde{\alpha}(k), \tilde{\beta}(k)\}$	Fourier transformed field

(b) TIME EVOLUTION

Next we have to evolve the eigenvalues λ and the scattering data S forward in t.

• AMAZING FACT :

$u(x,t)$ obeys KdV eqn $\Longrightarrow$ Eigenvalues $\lambda(t)$ are independent of t !

Therefore we only need to evolve the scattering data $S(0) \longrightarrow S(t)$, and this time evolution is simple.

(c) REASSEMBLY (INVERSE SCATTERING)

The final step is to reconstruct the potential $u(x,t)$ from the scattering data $S(t)$ at time t. This is called "inverse scattering".

It may be surprising that one can do it (cf. "Can you hear the shape of a drum?"), but that this is possible for these Schrödinger problems was already known at the time of GGKM.

• Understanding each of these 3 steps will take us time.
It will be a good idea to keep referring back to this overview as we go along.