8. THE BASICS OF SCATTERING THEORY

[DJ §3.1, 3.2]

- We need to analyse the possible solutions to

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + u\right)\psi = \lambda\psi} \quad \Leftrightarrow \quad \boxed{L\psi = \lambda\psi} \tag{8.1}$$

with ψ bounded $\forall x$. (This restricts the possible λ's.)

For this section t, which appears as a parameter through $u(x,t)$, is fixed. (and dropped from the notation)

8.1 - Physical interpretation

- FACT: the equation

$$\boxed{i\frac{\partial}{\partial \tau}\Psi(x,\tau) = \left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,\tau)} \tag{8.2}$$

TIME-DEPENDENT SCHRÖDINGER EQN

describes a particle (of mass $m=\frac{1}{2}$) subject to a potential $V(x)$ in quantum mechanics. $|\Psi(x,\tau)|^2 dx$ is the probability to find the particle in $[x, x+dx]$ at time τ.

NOTE: the QM time τ in (8.2) has nothing to do with the KdV time t!
(The KdV time t is fixed here and is only a parameter hidden in $V(x)$.)

- To solve (8.2), separate variables

$$\boxed{\Psi(x,\tau) = \psi(x)\,\phi(\tau)} \tag{8.3}$$

and sub in:

$$i\psi\dot{\phi} = (-\psi'' + V\psi)\phi \qquad \left(\dot{} = \frac{d}{d\tau}, \quad ' = \frac{d}{dx}\right)$$

$$\Rightarrow \quad i\,\underbrace{\frac{\dot{\phi}}{\phi}}_{\text{only depends on } \tau} = \underbrace{\frac{-\psi'' + V\psi}{\psi}}_{\text{only depends on } x} = \text{const} \equiv k^2 \qquad (8.4)$$

Therefore

$$\boxed{\phi(\tau) = e^{-ik^2\tau}} \qquad (8.5)$$

and $\psi(x)$ satisfies

$$\boxed{\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)} \qquad (8.6)$$

TIME-INDEPENDENT SCHRÖDINGER EQN

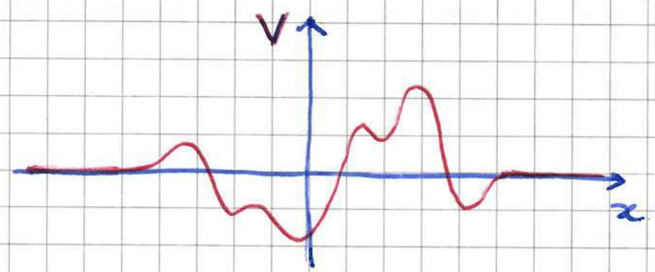
which is the same as our equation (8.1), with

KdV field $\rightarrow$ $u = -V$ $\leftarrow$ - Potential energy in QM

Eigenvalue of L $\rightarrow$ $\lambda = -k^2$ $\leftarrow$ - Total energy in QM (8.7)

In QM (8.6) describes a particle with energy $E = k^2$ ($= -\lambda$) in the potential $V(x)$.

- We consider potentials s.t. $V(x) \xrightarrow[x \to \pm\infty]{} 0$: (sufficiently fast)



- In CLASSICAL MECHANICS, a particle of energy E ($= T+V$) is localised and bounces off the potential at the turning points x_* where $V(x_*) = E$.
  ↳ reverses its direction of motion

→ EXAMPLES USING THE PICTURE.

- In QUANTUM MECHANICS, there is non-vanishing probability to find the particle anywhere (if $V<\infty$), and the particle can "tunnel" through potential barriers.

- The SCATTERING DATA will be encoded in the ASYMPTOTICS of ψ as $x \to \pm\infty$, where $V(x) \to 0$ and (8.6) is

$$-\frac{d^2}{dx^2}\psi = k^2\psi . \tag{8.8}$$

$\to$ 2 independent solutions $e^{\pm ikx}$.

So the general solution with eigenvalue $E=k^2$ has the asymptotics

$$\boxed{\begin{aligned} \psi(x) &\approx A(k)\,e^{ikx} + B(k)\,e^{-ikx} & (x\to-\infty) \\ \psi(x) &\approx C(k)\,e^{ikx} + D(k)\,e^{-ikx} & (x\to+\infty) \end{aligned}} \tag{8.9}$$

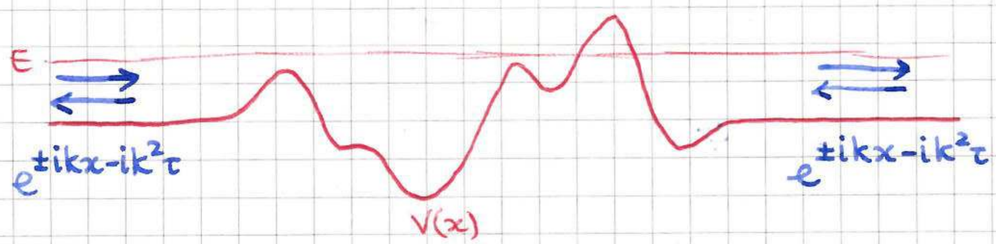
and restoring the τ-dependence

$$\boxed{\begin{aligned} \Psi(x,\tau) &\approx A(k)\,e^{ikx-ik^2\tau} + B(k)\,e^{-ikx-ik^2\tau} & (x\to-\infty) \\ \Psi(x,\tau) &\approx C(k)\,e^{ikx-ik^2\tau} + D(k)\,e^{-ikx-ik^2\tau} & (x\to+\infty) \end{aligned}} \tag{8.10}$$

RIGHT-MOVING WAVE

LEFT-MOVING WAVE

This is a bounded soln if $\boxed{E=k^2>0}$. (and we can take $k>0$ WLOG)



- As we will see in examples, finding the solution in the "interacting region" (where $V(x) \neq 0$) which interpolates between the 2 asymptotic regions imposes 2 relations among A, B, C, D, leaving 2 undetermined coefficients as is appropriate for a 2nd order ODE.

- To fix the remaining two coefficients, if $k^2 > 0$ we will impose

$$\boxed{A(k) = 1 \quad , \quad D(k) = 0} \tag{8.11}$$

and call

$$\boxed{\begin{aligned} B(k) &\equiv R(k) \\ C(k) &\equiv T(k) \,. \end{aligned}} \qquad \begin{array}{l} \text{REFLECTION COEFFICIENT} \\ \text{TRANSMISSION COEFFICIENT} \end{array} \tag{8.12}$$

The resulting "SCATTERING SOLUTION" has asymptotics

$$\boxed{\begin{aligned} \psi(x) &\approx e^{ikx} + R(k) e^{-ikx} \quad , \quad x \to -\infty \\ \psi(x) &\approx T(k) e^{ikx} \quad , \quad x \to +\infty \end{aligned}} \tag{8.13}$$

and represents a unit flux ($A(k)=1$) of incoming rightmoving particles (coming towards the potential from $x = -\infty$), which is partially reflected by the potential ($R(k)$) and partially transmitted through the potential ($T(k)$).

One can prove (* Ex 49)

$$\boxed{|R(k)|^2 + |T(k)|^2 = 1} \tag{8.14}$$

which is the statement of conservation of probability:

incoming particles are either reflected or transmitted.

Aside: instead of a (constant in time) unit flux of incoming particles, one can also consider a localized incoming wavepacket $\Psi(x,\tau)$ with approximate position and momentum. This is achieved by superimposing scattering solutions of the type (8.13) with different k, as we did for linear wave eqns at the beginning of last term.

CONCLUSION:
Solving (8.1) with $-\lambda = k^2 > 0$ is equivalent to analysing a QM scattering problem of a particle of energy k^2 moving in a potential $V(x) = -u(x)$.

- What about $\boxed{k^2 < 0}$?

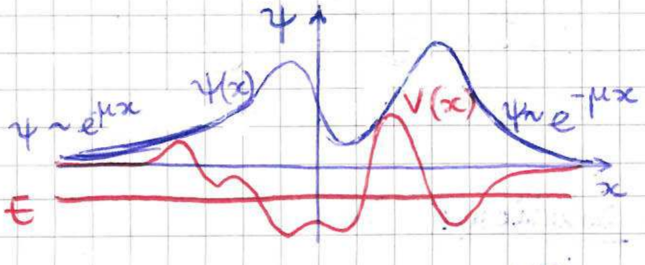
Letting $k = i\mu$ $(\mu > 0)$, so that $E = -\mu^2$, the asymptotics of the general solution (8.9) becomes

$$\boxed{\begin{aligned} \psi(x) &\approx a(\mu)\, e^{-\mu x} + b(\mu)\, e^{\mu x}, \quad x \to -\infty \\ \psi(x) &\approx c(\mu)\, e^{-\mu x} + d(\mu)\, e^{\mu x}, \quad x \to +\infty \end{aligned}} \tag{8.15}$$

and

$$\boxed{\psi \text{ is bounded} \iff a(\mu) = d(\mu) = 0} \,. \tag{8.16}$$

(possibly empty)



In this case $|\psi(x)|^2 \xrightarrow[x \to \pm\infty]{} 0$: "BOUND STATE SOLUTION".

- Given a potential V and eigenvalues $\lambda = -E = \mu^2 > 0$, bound state solutions will only exist for a (possibly empty) finite set of eigenvalues

$$\{\mu_k\}_{k=1}^{N} = \{\mu_1, \mu_2, \dots, \mu_N\} . \qquad (8.17)$$
$$\mu_1 < \mu_2 < \dots < \mu_N$$

- SUMMARY : OF THIS PREVIEW

1) $-\lambda = +k^2 \in (0, +\infty)$ "CONTINUOUS SPECTRUM" (or "CONTINUUM")

→ SCATTERING SOLUTIONS . (Oscillatory asymptotics)

2) $-\lambda = -\mu^2 \in \{-\mu_1^2, \dots, -\mu_N^2\}$ "DISCRETE SPECTRUM" (or "DISCRETUUM")

→ BOUND STATE SOLUTIONS. (Damped asymptotics)

8.2 - Examples

8.2.1 - $V(x) = 0$

- Already done when we looked at asymptotics. We want to solve

$$-\frac{d^2}{dx^2}\psi = k^2 \psi .$$

- 2 cases :

General solution

ⓐ $k^2 > 0$: $\psi = A e^{ikx} + B e^{-ikx}$ (8.18)

left/right-moving wave, bounded.

Scattering solution :

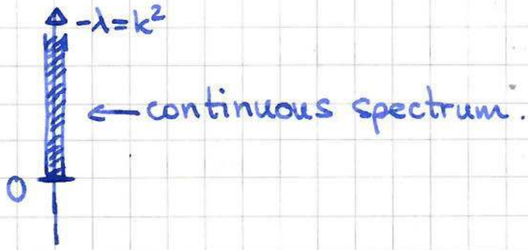
$$\psi = e^{ikx} \qquad (8.18)$$

$$\Rightarrow R(k) = 0, \quad T(k) = 1 . \qquad (8.19)$$

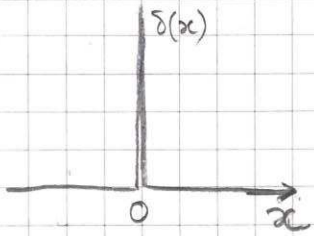
(b) $\underline{k^2 = -\mu^2 < 0}$: $\quad \psi = a e^{-\mu x} + b e^{\mu x}$ (8.20)
$(\mu > 0)$

The only way to avoid $\psi \to \infty$ as $x \to \pm\infty$ is $a = b = 0$.
$\to$ No bound state solutions.

- Conclusion: if $u = 0$, $L\psi = \lambda\psi = -k^2\psi$ has a solution $\forall \lambda \underset{(-)}{<} 0$ and no solutions for $\lambda > 0$.



$\boxed{8.2.2 - \quad V(x) = a\,\delta(x)}$



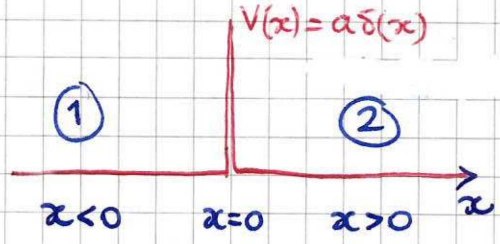
Recall: $\delta(x) = 0$ for $x \neq 0$, $\displaystyle\int_{-\infty}^{+\infty} dx\, f(x)\,\delta(x) = f(0)$.

We can think of $\delta(x)$ as the "limit" of a sequence of more and more peaked/concentrated (even) functions centred at 0 and with unit integral.

E.g.
$$\frac{1}{\sqrt{\pi a}}\, e^{-x^2/a} \;\; \text{"}\!\!\xrightarrow[a \to 0^+]{}\!\!\text{"} \;\; \delta(x) \qquad \text{or} \qquad \begin{cases} \frac{1}{2a}, & |x| < a \\ 0, & |x| > a \end{cases} \;\; \text{"}\!\!\xrightarrow[a \to 0^+]{}\!\!\text{"} \;\; \delta(x)$$

What we mean is that
$$\frac{1}{\sqrt{\pi a}}\, e^{-\frac{x^2}{a}} \xrightarrow[a \to 0^+]{} 0 \quad \text{for } x \neq 0, \qquad \int_{-\infty}^{+\infty} dx\, f(x) \frac{1}{\sqrt{\pi a}}\, e^{-\frac{x^2}{a}} \xrightarrow[a \to 0^+]{} f(0)$$

and similarly for other sequences of functions.



(if $a > 0$)

ⓐ $k^2 \gtrless 0$: General solution

$$\psi = \begin{cases} A^{(k)} e^{ikx} + B^{(k)} e^{-ikx}, & x<0 \quad ① \\ C^{(k)} e^{ikx} + D^{(k)} e^{-ikx}, & x>0 \quad ② \end{cases} \tag{8.21}$$

since $V=0$ for $x \neq 0$.

- To finish, we must match the 2 parts of the solution (8.21) at $x=0$.
  To find the matching conditions, let's integrate the time-indep. Schrödinger equation

$$-\psi''(x) + a\,\delta(x)\,\psi(x) = k^2 \psi(x) \tag{8.22}$$

in an infinitesimal neighbourhood $[-\varepsilon, \varepsilon]$ of $x=0$:

$$\int_{-\varepsilon}^{\varepsilon} dx \left[-\psi''(x) + a\,\delta(x)\,\psi(x)\right] = k^2 \int_{-\varepsilon}^{\varepsilon} dx\, \psi(x)$$

$$\Rightarrow -\left[\psi'(x)\right]_{-\varepsilon}^{\varepsilon} + a\,\psi(0) = k^2 \int_{-\varepsilon}^{\varepsilon} dx\, \psi(x) \tag{8.23}$$

If ψ is bounded, taking $\varepsilon \to 0$ gives

$$\boxed{-\left[\psi'(x)\right]_{0^-}^{0^+} + a\,\psi(0) = 0} \tag{8.24}$$

← (Discontinuity of ψ' at 0) $= a\,\psi(0)$.

Similarly, one can integrate (8.22) twice to get

$$\boxed{\left[\psi(x)\right]_{0^-}^{0^+} = 0}\,. \tag{8.25}$$

← ψ is continuous at 0

Applying the matching conditions (8.25) to (8.21):

$$\begin{cases} A + B = C + D \\ ik(C-D) - ik(A-B) = a(A+B) = a(C+D) \end{cases}$$

$$\Leftrightarrow \begin{cases} A + B = C + D \\ A - B = \left(1 - \frac{a}{ik}\right)C - \left(1 + \frac{a}{ik}\right)D \end{cases}$$

$$\Leftrightarrow \begin{cases} A = \left(1 - \frac{a}{2ik}\right)C - \frac{a}{2ik}D \\ B = \frac{a}{2ik}C + \left(1 + \frac{a}{2ik}\right)D \end{cases} \qquad (8.26)$$

Subbing (8.26) in (8.21) gives the general solution. (2 integr. constants)

- Scattering solution: set $D = 0$ and normalize so that $A = 1$.
  (This is achieved by replacing ψ by $\frac{\psi}{A}$. Still a sol'n because eqn is linear.)

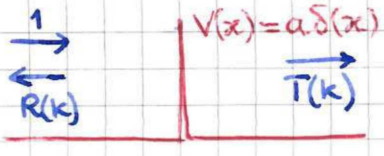
$$\boxed{\psi(x) = \begin{cases} e^{ikx} + \underbrace{\frac{a}{2ik-a}}_{=R(k)} e^{-ikx}, & x < 0 \\ \underbrace{\frac{2ik}{2ik-a}}_{=T(k)} e^{ikx}, & x > 0 \end{cases}} \qquad (8.27)$$

Therefore

$$\boxed{\begin{aligned} R(k) &= \frac{a}{2ik-a} \\ T(k) &= \frac{2ik}{2ik-a} \end{aligned}} \qquad (8.28)$$

REFLECTION COEFFICIENT

TRANSMISSION COEFFICIENT



It's easy to check using (8.28) that

$$\boxed{|R(k)|^2 + |T(k)|^2 = 1} \, . \tag{8.29}$$

Prob. that incoming particle is reflected by the potential barrier

Prob that incoming particle is transmitted by the potential barrier.

(b) $\underline{k^2 = -\mu^2 < 0}$ $\quad (\mu > 0)$:

We can obtain the general solution by setting $\underline{k = i\mu}$ in (8.21), (8.26):

$$\psi(x) = \begin{cases} A(i\mu)\, e^{-\mu x} + B(i\mu)\, e^{\mu x} \quad , & x < 0 \\ C(i\mu)\, e^{-\mu x} + D(i\mu)\, e^{\mu x} \quad , & x > 0 \end{cases} \tag{8.30}$$

This is <u>bounded</u> for $\underline{x \to \pm\infty}$ iff

$$\boxed{A(i\mu) = D(i\mu) = 0} \, . \tag{8.31}$$

Sub in (8.26):

$$\begin{cases} \left(1 + \frac{a}{2\mu}\right) C = 0 \\ B = -\frac{a}{2\mu} C \end{cases} .$$

Two options:

1) $C = B \; (= A = D) = 0$ trivial

2) $$\boxed{\mu = -\frac{a}{2} \, , \quad B = C} \, . \tag{8.32}$$

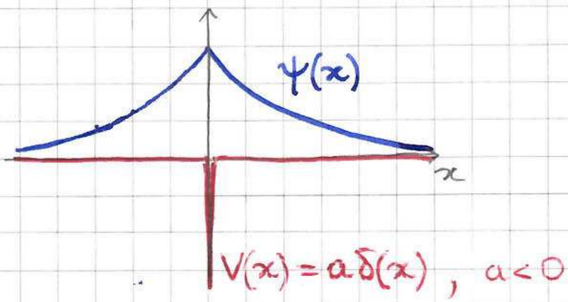
Option 2) is an acceptable bounded solution $(\mu > 0)$ only if $\underline{a < 0}$.

Therefore a <u>bound state solution</u>

$$\boxed{\psi(x) = \begin{cases} e^{-\frac{a}{2}x}, & x<0 \\ e^{\frac{a}{2}x}, & x>0 \end{cases} = e^{\frac{a}{2}|x|}} \qquad (8.33)$$

exists if $\underline{a<0}$ for $\underline{\mu = -a/2}$, that is for

$$\boxed{k^2 = -\frac{a^2}{4}} \qquad (8.34)$$



• <u>NOTE</u>: we need $D=0$ for the solution to be bounded as $x \to +\infty$.
Hence we could substitute $k = i\mu$ directly in the scattering sol'n (8.27)

$$\Rightarrow \psi(x) = \begin{cases} e^{-\mu x} + \dfrac{a}{-2\mu - a} e^{\mu x}, & x<0 \\ \dfrac{-2\mu}{-2\mu - a} e^{-\mu x}, & x>0 \end{cases} \qquad (8.35)$$

which looks hopelessly unbounded as $x \to -\infty$ due to the $e^{-\mu x}$ term.

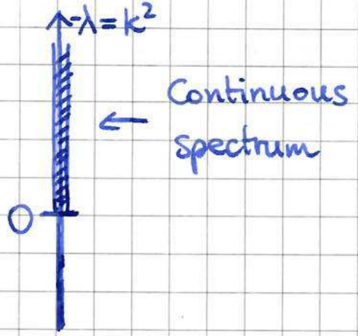
<u>Trick</u>: change normalization dividing (8.35) by $T(i\mu) = \dfrac{2\mu}{2\mu + a}$, to get

$$\psi(x) = \begin{cases} \dfrac{2\mu + a}{2\mu} e^{-\mu x} - \dfrac{a}{2\mu} e^{\mu x}, & x<0 \\ e^{-\mu x}, & x>0 \end{cases} \qquad (8.36)$$

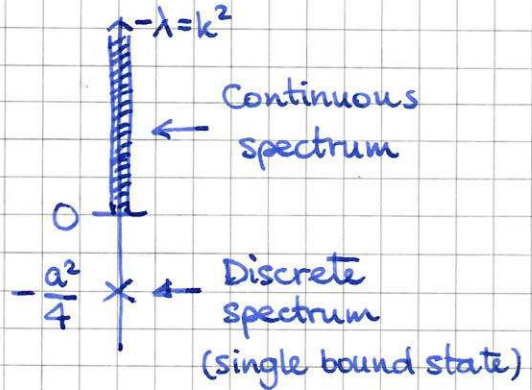
This solution is now bounded at $x=-\infty$ if $\mu=-\frac{a}{2}$, in which case we recover (8.33).

- **SUMMARY FOR $V(x)=a\,\delta(x)$**

$\underline{a \geq 0}$:



$\underline{a<0}$:



- For $\underline{\text{all}}$ $k^2>0$, a $\underline{\text{scattering solution}}$

$$\boxed{\psi(x)=\begin{cases} e^{ikx}+R(k)\,e^{-ikx}, & x<0 \\ T(k)\,e^{ikx}, & x>0 \end{cases}} \qquad (8.37)$$

exists with reflection and transmission coefficients

$$R(k)=\frac{a}{2ik-a}, \qquad T(k)=\frac{2ik}{2ik-a}.$$

- For $\underline{\text{isolated}}$ $k^2=-\mu^2<0$, a $\underline{\text{bound state solution}}$

$$\boxed{\psi(x)=\begin{cases} \dfrac{R(i\mu)}{T(i\mu)}\,e^{\mu x}, & x<0 \\ e^{-\mu x}, & x>0 \end{cases}} \qquad (8.38)$$

exists if $\underline{\mu=-\frac{a}{2}}^{>0}$, s.t.

$$\boxed{\frac{1}{T(i\mu)}=0} = \frac{1}{R(i\mu)} \qquad (8.39)$$

- **Bound state solutions** are obtained from the **scattering solution** upon

  1) dividing through by $T(k)$

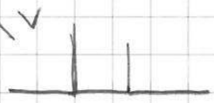
  2) setting

$$\boxed{k = i\mu = \text{pole of } T(k) \text{ on the positive imaginary axis}}$$

- **GENERAL STORY:**

$$\boxed{\text{Discrete spectrum of } -\frac{d^2}{dx^2}+V(x)} \Leftrightarrow \boxed{\text{Eigenvalues } -\lambda = k^2 < 0} \Leftrightarrow \boxed{\text{Bound states}} \Leftrightarrow \boxed{\text{Poles of } T(k) \text{ on the +ve imaginary axis}}$$

For more examples, see Ex 50 & Ex 51.



8.3 – Reflectionless potentials

- Return to the initial values $u(x,0) = \underset{\text{real constant}}{a}\,\text{sech}^2(x)$ for the KdV field considered at the beginning of chapter 7. Then the potential is

$$\boxed{V(x) = -a\,\text{sech}^2 x} \tag{8.40}$$

and we wish to solve the SE

$$\boxed{-\psi''(x) - a\,\text{sech}^2 x \cdot \psi(x) = k^2\psi(x)}\,. \tag{8.41}$$

- Substitute

$$\boxed{y = \tanh x} \quad \in (-1,1) \,, \tag{8.42}$$

so that

$$\frac{d}{dx} = \operatorname{sech}^2 x \, \frac{d}{dy} = (1-y^2)\frac{d}{dy} \,. \tag{8.43}$$

The TISE (8.41) becomes

$$\boxed{\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left(\frac{k^2}{1-y^2} + a\right)\psi = 0} \,. \tag{8.44}$$

Putting

$$\boxed{k^2 = -m^2 \,, \quad a = n(n+1)} \,, \tag{8.45}$$

(8.44) becomes

$$\boxed{\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left[n(n+1) - \frac{m^2}{1-y^2}\right]\psi = 0} \,. \tag{8.46}$$

GENERAL (or ASSOCIATED) LEGENDRE EQUATION

- The solutions of this equation are known.

- **FACT 1:** If $\underline{n = 0,1,2,\ldots}$ ($\Leftrightarrow n \in \mathbb{Z}_{\geq 0}$) and $\underline{m=0}$ ($\Rightarrow \underline{k=0}$), (8.46) becomes the Legendre equation and its bounded solution for $y \in [-1,1]$ is

$$\psi = P_n(y) = \frac{1}{n!\,2^n}\left(\frac{d}{dy}\right)^n (y^2-1)^n \tag{8.47}$$

n^{th} LEGENDRE POLYNOMIAL of the 1^{st} KIND

E.g.

$$P_1(y) = y$$

$$P_2(y) = -\frac{1}{2} + \frac{3}{2}y^2$$

$$P_3(y) = -\frac{3}{2}y + \frac{5}{2}y^3$$

$$\vdots$$

(The 2^{nd} solution is the Legendre function of the 2^{nd} kind $Q_n(y)$, which $\xrightarrow[y\to\pm1]{} \infty$.)

- **FACT 2:** (we'll derive it below)

If $n \in \mathbb{Z}_{\geq 0}$, bounded solutions to (8.46) only exist for

$$\boxed{m = 0, 1, 2, \dots, n} \tag{8.48}$$

and are

$$\boxed{P_n^m(y) = (-1)^m (1-y^2)^{\frac{m}{2}} \left(\frac{d}{dy}\right)^m P_n(y)} \tag{8.49}$$

ASSOCIATED LEGENDRE "POLYNOMIALS" of 1^{st} KIND

(" " because it's only a polynomial if m is even.)

- **FACT 3:**

Even when m, n are not integer (and in fact they could even be complex)

$$\boxed{P_n^m(y) = \frac{1}{\Gamma(1-m)} \left(\frac{1+y}{1-y}\right)^{m/2} {}_2F_1\left(-n, n+1; 1-m; \frac{1-y}{2}\right)} \tag{8.50}$$

solves (8.46), and reduces to (8.49) if $n \in \mathbb{Z}_{\geq 0}$ and $m = 0, 1, \dots, n$.

Here

$$\boxed{\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}} \tag{8.51}$$

EULER's GAMMA FUNCTION

which satisfies

$$\Gamma(N+1) = N! \qquad \text{if } N \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\} \tag{8.52}$$

$$\Gamma(z) \neq 0 \ \forall z, \quad \frac{1}{\Gamma(z)} = 0 \qquad \text{iff } z \in \{0, -1, -2, \dots\} \tag{8.53}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{8.54}$$

$_2F_1$ is the HYPERGEOMETRIC FUNCTION, which has the Taylor expansion

$$\boxed{{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma(k+c)} \frac{z^k}{k!}} \tag{8.55}$$

if $|z|<1$.

[This is quite a bit of technology, but bear with me: we'll learn something interesting!]

• So, up to normalization, the (potentially) bounded solution to (8.44) is

$$\boxed{\psi = P_n^m(y)} \tag{8.56}$$

with

$$\boxed{m = ik \quad , \quad n = \frac{\sqrt{1+4a}}{2} - \frac{1}{2}} \tag{8.57}$$

(I picked the roots which give the scattering sol'n with particles incident from the left)

Note: $n \geq 0$ if $a \geq 0$.

$n \geq -\frac{1}{2}$ and real if $a \geq -\frac{1}{4}$.

(a) $k^2 > 0$ – CONTINUOUS SPECTRUM

• $x \to +\infty$:

$$y = \tanh x \approx 1 - 2e^{-2x} \longrightarrow 1^-$$

$$F\left(\ldots ; \frac{1-y}{2}\right) \longrightarrow 1$$

$$\frac{1+y}{1-y} \approx e^{2x}$$

$$\Rightarrow \quad \psi \approx \frac{1}{\Gamma(1-ik)} e^{ikx} \quad \text{as } x \to +\infty \tag{8.58}$$

• $x \to -\infty$:

$$y \approx -1 + 2e^{2x} \longrightarrow -1^+$$

$$\frac{1+y}{1-y} \approx e^{2x}$$

and it turns out that

$$\frac{1}{\Gamma(1-m)} {}_2F_1\left(-n, n+1; 1-m; \frac{1-y}{2}\right) \approx \frac{\Gamma(-m)}{\Gamma(1-m+n)\Gamma(-m-n)} + \frac{\Gamma(m)}{\Gamma(-n)\Gamma(n+1)} e^{-2mx} .$$

[This asymptotics follows from the identity

$$\frac{\sin(\pi(c-a-b))}{\pi}\,{}_2F_1(a,b;c;z) = \frac{1}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c+1)}\,{}_2F_1(a,b;a+b-c+1;1-z)$$
$$-\frac{(1-z)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}\,{}_2F_1(c-a,c-b;c-a-b+1;1-z).\ ]$$

$$\Rightarrow \psi \approx \frac{\Gamma(-ik)}{\Gamma(1-ik+n)\Gamma(-ik-n)}e^{ikx} + \frac{\Gamma(ik)}{\Gamma(-n)\Gamma(n+1)}e^{-ikx} \quad \text{as } \underline{x\to-\infty}. \tag{8.59}$$

- Normalizing this scattering solution so that there's a unit flux of incoming particles from the left, we find the reflection/transmission coefficients

$$\boxed{\begin{aligned} R(k) &= \frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)\Gamma(1+n)\Gamma(-n)} = -\frac{\sin(\pi n)}{\pi}\cdot\frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)} \\ T(k) &= \frac{\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(1-ik)\Gamma(-ik)} \end{aligned}} \tag{8.60}$$

- <u>NOTE</u>: $\underline{R(k) = 0 \quad \forall k>0}$ if $\underline{n}$ is an <u>integer</u>!

  The corresponding potentials

$$\boxed{V(x) = -n(n+1)\,\mathrm{sech}^2 x} \tag{8.61}$$

  with $\underline{n\in\mathbb{Z}_{\geq 0}}$ (wlog) are called <u>REFLECTIONLESS</u>:
  no particles are reflected, for <u>any</u> wavenumber k!

We'll see later that reflectionless potentials are associated to KdV solutions which contain solitons and no dispersive waves.

(b) $k^2 < 0$ - DISCRETE SPECTRUM

- Set $k = i\mu$ and divide through by $T(i\mu)$:

$$\psi \approx \begin{cases} \dfrac{1}{T(i\mu)} e^{-\mu x} + \dfrac{R(i\mu)}{T(i\mu)} e^{\mu x}, & x \to -\infty \\ e^{-\mu x}, & x \to +\infty \end{cases} \tag{8.62}$$

This solution is bounded (at $x = -\infty$) iff $\dfrac{1}{T(i\mu)} = 0$ for some $\mu > 0$.

$$\frac{1}{T(i\mu)} = \frac{\Gamma(1+\mu)\Gamma(\mu)}{\Gamma(1+\mu+n)\Gamma(\mu-n)} = 0$$

$$\Longleftrightarrow \quad \underset{j \in \mathbb{Z}_{\geq 0}}{1+\mu+n = -j} \quad (1) \qquad \vee \qquad \underset{h \in \mathbb{Z}_{\geq 0}}{\mu - n = -h} \quad (2)$$

(Recall (8.53): $\Gamma(z)$ has no zeros, and has simple poles at $-z \in \mathbb{Z}_{\geq 0}$.)

- If $n \notin \mathbb{R} \Leftrightarrow a < -\frac{1}{4}$, there are no real solutions for μ.
- If $n \in \mathbb{R}$, we can take $n \geq -\frac{1}{2}$ wlog.
  Then solutions exist for positive μ from option (2)

$$\boxed{\mu = n,\ n-1,\ n-2,\ \dots,\ n - \lfloor n \rfloor} \,. \tag{8.63}$$

$\lfloor n \rfloor$ = "floor of n" = (Largest integer $\leq n$)

So

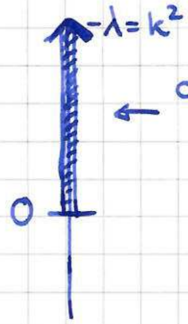
$$\boxed{\begin{matrix}\text{Total number} \\ \text{of bound states}\end{matrix} = \lceil n \rceil} \tag{8.64}$$

$$= (\text{Smallest integer} \geq n)$$

(If n is an integer, the last eigenvalue $\mu = 0$ must be discarded: NOT A BOUND STATE!)

- **SUMMARY** for $V(x) = -a\,\mathrm{sech}^2 x \equiv -n(n+1)\,\mathrm{sech}^2 x$:

$a<0$:



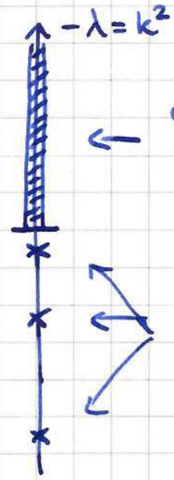
continuous spectrum

$R(k) \neq 0$

$a = n(n+1) \geq 0$:

1) n not integer:

(say $n = 2.5$)



continuous spectrum

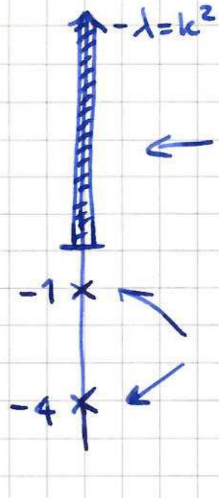
$R(k) \neq 0$

discrete spectrum

$\lceil n \rceil$ bound states

2) $n \in \mathbb{Z}_{\geq 0}$:

(say $n=2$)



continuous spectrum

discrete spectrum

n bound states .