9. EVOLVING THE SCATTERING DATA

[DJ §5.2]

9.1 - Scattering data for general potentials

- SO FAR: for any localised INITIAL DATA $u(x,0)$ for KdV, the auxiliary time-independent Schrödinger eqn

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x) \tag{9.1}$$

with potential $V(x) = -u(x,0)$ has:

1) CONTINUOUS SPECTRUM of $E = k^2 \geq 0$ eigenvalues.

eigenfns
$$\psi_k(x) \approx \begin{cases} e^{ikx} + R(k)\,e^{-ikx} & , \ x \to -\infty \\ T(k)\,e^{ikx} & , \ x \to +\infty \end{cases} \tag{9.2}$$

Normalization: incoming flux = 1.

2) DISCRETE SPECTRUM of $E = k^2 = -\mu^2 < 0$ eigenvalues.

eigenfns
$$\psi_n(x) \approx \begin{cases} c_n\,e^{\mu_n x} & , \ x \to -\infty \\ d_n\,e^{-\mu_n x} & , \ x \to +\infty \end{cases} \tag{9.3}$$

Normalization: $\langle \psi_n, \psi_n \rangle \equiv \int_{-\infty}^{+\infty} dx\, |\psi_n(x)|^2 = 1$.

- Luckily we don't need to find the explicit form of the eigenfunctions for a general potential $V(x) = -u(x)$.
It is enough to know the EIGENVALUES + limited info on the ASYMPTOTICS OF THE EIGENFUNCTIONS: (at $x \to -\infty$)

$$\boxed{\{R(k), \{\mu_n, c_n\}\}} \quad \text{"SCATTERING DATA"} \tag{9.4}$$

- Clearly, u (or $V=-u$) determines completely the scattering data.
  (This was step ⓐ $\downarrow$: "DISASSEMBLY" / "SCATTERING".)

- Amazingly, the converse is also true: u (or $V=-u$) can be reconstructed ("reassembled") completely from the scattering data.
  (This will be step ⓒ $\uparrow$: "REASSEMBLY" / "INVERSE SCATTERING".)

- Our NEXT GOAL will then be to understand the intermediate step ⓑ $\longrightarrow$: TIME EVOLUTION OF THE SCATTERING DATA.

9.2 - Examples of scattering data

1) $V(x) = a\,\delta(x)$:

$$R(k) = \frac{a}{2ik-a}$$

Bound state: $\psi(x) = A e^{-\mu|x|}$ for $\mu = -\frac{a}{2} > 0$ (and $A \in \mathbb{R}$)

Normalization: $1 = \int_{-\infty}^{+\infty} dx\, |\psi(x)|^2 = 2A^2 \int_0^\infty dx\, e^{-2\mu x} = \frac{A^2}{\mu}$

$\Rightarrow$ Normalized bound state:

$$\psi(x) = \sqrt{-\frac{a}{2}}\; e^{\frac{a}{2}|x|}$$

$\Rightarrow$ Scattering data at $t=0$:

$$\boxed{\begin{array}{ll} \left\{R(k) = \frac{a}{2ik-a}\right\} & \text{if } a \geq 0 \\ \left\{R(k) = \frac{a}{2ik-a}, \left\{\mu_1 = -\frac{a}{2},\ c_1 = \sqrt{-\frac{a}{2}}\right\}\right\} & \text{if } a < 0 \end{array}} \qquad (9.5)$$

2) $V(x) = -n(n+1)\,\mathrm{sech}^2 x \quad , \; n \in \mathbb{Z}_{\geq 0}$ (REFLECTIONLESS POTENTIAL)

$$R(k) = 0$$

Bound states: $\psi_m(x) = A\, P_n^m(\tanh x) \quad , \; m = 1, \dots, n.$ ($A \in \mathbb{R}$ wlog)

(Note: $P_n^{-m}(y) \propto P_n^m(y)$, so I'll use $P_n^m(y)$ wlog.) E29

Normalization: $1 = \int_{-\infty}^{\infty} dx\, |\psi_m(x)|^2 = A^2 \int_{-1}^{1} \frac{dy}{1-y^2} P_n^m(y)^2 = A^2 \frac{(n+m)!}{m\,(n-m)!}$

(e.g. see Wikipedia. Note that $P_n^0(\tanh x)$ is not normalizable: $\int \dots = \infty$.)

Asymptotics: $P_n^m(\tanh x) \approx (-1)^n \frac{(n+m)!}{m!\,(n-m)!} e^{mx} \quad , \quad x \to -\infty$

(This takes some work ...)

$\Rightarrow$ Asymptotics of normalized bound state:

$$\psi_m(x) \approx (-1)^n \frac{1}{m!} \sqrt{\frac{m\,(n+m)!}{(n-m)!}}\, e^{mx} \quad , \qquad x \to -\infty .$$

$\Rightarrow$ Scattering data at $t=0$:

$$\boxed{\left\{ R(k) = 0 \; , \quad \left\{ \mu_m^{(n)} = m, \; c_m^{(n)} = (-1)^n \frac{1}{m!} \sqrt{\frac{m\,(n+m)!}{(n-m)!}} \right\}_{m=1}^{n} \right\}} \qquad (9.6)$$

3) $V(x) = -n'(n'+1)\,\mathrm{sech}^2 x\,, \quad n' = n+\varepsilon\,, \; |\varepsilon| \ll 1$ (ALMOST REFLECTIONLESS POTENTIAL)

SKIP

LEAVE AS READING MATERIAL

This is a small perturbation of the previous case.

The discrete eigenvalues are $\mu' = (\varepsilon,)\, 1+\varepsilon, 2+\varepsilon, \dots, n+\varepsilon$.

This new eigenvalue is there only if $\varepsilon > 0$. Let's assume $\varepsilon < 0$ and not worry about it

We just need to replace factorials by Gamma functions:

$$\Gamma(n') = \Gamma(n)\left(1 + \varepsilon\psi(n) + O(\varepsilon^2)\right) \tag{9.7}$$

where

$$\boxed{\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}} \qquad \text{DIGAMMA FUNCTION}\,. \tag{9.8}$$

Expanding to 1st order in ε one finds

$$\boxed{\underbrace{c_m^{(n')}}_{\text{new}} \approx \underbrace{c_m^{(n)}}_{\text{old, see (9.6)}} \cdot \left[1 + \varepsilon\left(\frac{1}{m} + \frac{\psi(n+m+1)}{2} - \psi(m+1)\right)\right]} \tag{9.9}$$

for the discrete eigenvalues $\boxed{\mu_m^{(n')} = m+\varepsilon}\,, \quad m = 1, \dots, n\,.$ (9.10)

The reflection coefficient is easier and more interesting. From (8.60)

$$\boxed{\begin{aligned} R^{(n')}(k) &= -\frac{\sin(\pi(n+\varepsilon))}{\pi}\,\frac{\Gamma(ik)\Gamma(1-ik+n+\varepsilon)\Gamma(-ik-n-\varepsilon)}{\Gamma(-ik)} \\ &\approx \varepsilon \cdot (-1)^{1+n}\,\frac{\Gamma(ik)\Gamma(1-ik+n)\Gamma(-ik-n)}{\Gamma(-ik)} \end{aligned}} \tag{9.11}$$

(9.9) - (9.11) are the new scattering data.

Next: we want to understand how to evolve the scattering data forward in (KdV) time, when $u = -V$ evolves according to the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

9.3 – The idea of a Lax pair

- We want to solve the initial value problem for a PDE

$$\boxed{u_t = N(u)} \qquad \text{with } u, u_x, u_{xx}, \ldots \xrightarrow[x\to\pm\infty]{} 0 \tag{9.12}$$

where $N(u)$ is a function of $u, u_x, u_{xx}, \ldots$ (but no t-derivatives).
For KdV $N(u) = -6uu_x - u_{xxx}$, but we can be more general.

- We related it to the eigenvalue problem

$$\boxed{L(u)\,\psi = \lambda\psi} \tag{9.13}$$

where

$$\boxed{L(u) = \frac{\partial^2}{\partial x^2} + u(x,t)} \tag{9.14}$$

- $L(u)$ depends on t through u. Then we'd expect the eigenfunctions ψ and the eigenvalues λ of $L(u)$ to depend on t too. But in fact:

THEOREM:

If $\exists$ a differential operator $\underline{M(u)}$ s.t.

$$\underset{\text{(e.g. KdV eqn)}}{\boxed{u_t = N(u)}} \iff \boxed{L(u)_t = [M(u), L(u)]} \quad \text{LAX EQUATION} \tag{9.15}$$

(i) then the SPECTRUM of $L(u)$ (= set of eigenvalues of $L(u)$) is independent of t, and

(ii) there is a set of eigenfunctions ψ of $L(u)$ which evolves in t as

$$\boxed{\psi_t = M(u)\,\psi}\,. \tag{9.16}$$

(In fact, all bound states and scattering solutions of (9.13) evolve according to (9.16) for some $M(u)$, if the Lax eqn (9.15) is satisfied.)

(i) is very striking. It means that the spectra of $\frac{\partial^2}{\partial x^2}+u(x,0)$ and of $\frac{\partial^2}{\partial x^2}+u(x,t)$ are the same $(\forall t)$, which is totally unexpected.
(The operators are said to be "ISOSPECTRAL".)

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Def: $[A,B] \equiv AB - BA$ "COMMUTATOR" of A and B.

In our case $M(u)$, $L(u)$ are differential operators which involve x-derivatives. They need not commute. E.g.

$$\frac{\partial}{\partial x} g(x) \neq g(x) \frac{\partial}{\partial x} \qquad (*)$$

where the differential operators act on everything to their right, and the operator eqn means

$$L(u)_t f = [M(u), L(u)] f \qquad \forall \text{ functions } f.$$

For the example (*)

$$\frac{\partial}{\partial x}(g f) \neq g\left(\frac{\partial}{\partial x} f\right).$$

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• PROOF of (i)

$$L\psi = \lambda\psi$$

$$\underset{\frac{\partial}{\partial t}}{\Rightarrow} \quad L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t$$

$$\underset{\text{Lax eqn}}{\Rightarrow} \quad \lambda_t\psi = (ML - LM)\psi + (L-\lambda)\psi_t$$

$$= (M\lambda - LM)\psi + (L-\lambda)\psi_t$$

(λ is a number $\Rightarrow M\lambda = \lambda M$)

$$= (L-\lambda)(\psi_t - M\psi) \qquad (9.17)$$

Now consider

$$\boxed{\langle \psi_1, \psi_2 \rangle \equiv \int_{-\infty}^{+\infty} dx\, \overline{\psi_1}(x)\, \psi_2(x)}$$ HERMITIAN INNER PRODUCT (9.18)

(on square integrable functions L^2: $\langle \psi_i, \psi_i \rangle < \infty$) → Bound states

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• **LEMMA**: L is HERMITIAN (/SELF-ADJOINT), i.e.

$$\boxed{\langle \psi_1, L\psi_2 \rangle = \langle L\psi_1, \psi_2 \rangle} \quad \forall\, \psi_1, \psi_2 \in L^2 \tag{9.19}$$

Proof:

$$\langle \psi_1, L\psi_2 \rangle = \int_{-\infty}^{\infty} dx\, \overline{\psi_1}(x) \left(\frac{\partial^2}{\partial x^2} + u(x,t) \right) \psi_2(x)$$

Integrate by parts twice (no boundary terms because $\psi_i, \psi_i' \xrightarrow[x\to\pm\infty]{} 0$), and $u \in \mathbb{R}$

$$= \int_{-\infty}^{+\infty} dx \overline{\left[\left(\frac{\partial^2}{\partial x^2} + u(x,t) \right) \psi_1(x) \right]}\, \psi_2(x) = \langle L\psi_1, \psi_2 \rangle .$$

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Then (9.17) implies

$$\lambda_t \langle \psi, \psi \rangle = \langle \psi, (L-\lambda)(\psi_t - M\psi) \rangle$$

(Hermiticity of L, $\lambda \in \mathbb{R}$)

$$= \langle \underbrace{(L-\lambda)\psi}_{=0}, \psi_t - M\psi \rangle = 0 .$$

$$\Rightarrow \boxed{\lambda_t = 0} . \tag{9.20}$$

NOTE: the derivation holds for the discrete spectrum, so that the inner product is finite.
But we know already that the continuous spectrum is all $\lambda \leq 0$ for all t, so that part is trivial.

This proves (i).

- PROOF of (ii)

We need to show that $(L-\lambda)\psi = 0$ continues to hold at later times if ψ evolves according to $\psi_t = M\psi$.

$$\frac{\partial}{\partial t}\left((L-\lambda)\psi\right) = L_t\psi + L\psi_t - \underset{0}{\underset{\shortparallel}{\lambda_t}}\psi - \lambda\psi_t$$

$$\underset{(\psi_t = M\psi)}{=} L_t\psi + LM\psi - \lambda M\psi$$

$$\underset{\substack{\lambda M = M\lambda \\ L\psi = \lambda\psi}}{=} L_t\psi + LM\psi - ML\psi$$

$$= \left(L_t + [L,M]\right)\psi \qquad (9.21)$$

So if ψ starts off an eigenfunction and $L_t = [M,L]$, then ψ remains an eigenfunction; because integrating (9.21) wrt t gives

$$(L-\lambda)\psi = 0 \qquad \forall\, t. \qquad (9.22)$$

This proves (ii).

- L, M are called a "LAX PAIR".

The t-dependence of ψ has now been sorted out, so long as we can find M such that the equivalence (9.15) holds.

9.3.1 – The Lax pair for KdV

We have $L(u) = \frac{\partial^2}{\partial x^2} + u$, where u evolves according to KdV: $u_t + 6uu_x + u_{xxx} = 0$. We want to find $M(u)$ s.t.

$$\begin{array}{c} u_t + 6uu_x + u_{xxx} = 0 \\ \Updownarrow \\ u_t = N(u) \equiv -6uu_x - u_{xxx} \end{array} \iff \underset{u_t}{\underset{\shortparallel}{L(u)_t}} = [M(u), L(u)] \qquad (9.23)$$

Hence we need

$$[M(u), L(u)] = N(u) \equiv -6uu_x - u_{xxx} \tag{9.24}$$

- One can be systematic (more later), but let's first check that

$$\boxed{M(u) = -\left(4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x\right)} \tag{9.25}$$

does the job for KdV.

- SOME USEFUL PROPERTIES OF $D \equiv \frac{\partial}{\partial x}$:

$$[D,u]f = D(uf) - uDf = u_x f + u f_x - u f_x = u_x f \iff [D,u] = u_x \tag{9.26}$$

$$[D^n, u]f = D^n(uf) - uD^n f = \sum_{m=0}^{n} \binom{n}{m} (D^{n-m}u)(D^m f) \quad - uD^n f$$

$$= \sum_{m=0}^{n-1} \binom{n}{m} (D^{n-m}u)(D^m f) \quad ,$$

← Like derivative D^n of uf, but drop the last term!

that is

$$\boxed{[D^n, u] = \sum_{m=0}^{n-1} \binom{n}{m} u_{\underbrace{x\ldots x}_{n-m\,\text{times}}} D^m} . \tag{9.27}$$

Also, $[D^n, D^m] = 0$.

Finally, $\forall$ A, B, C

$$\begin{aligned} [A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA \\ &= [A,B]C + B[A,C], \end{aligned} \tag{9.28}$$

and similarly

$$[AB, C] = A[B,C] + [A,C]B . \tag{9.29}$$

• Now the actual calculation:

$$[L(u), M(u)] = [4D^3 + 6uD + 3u_x, D^2 + u]$$

$$= 4[D^3, u] + 6[uD, D^2] + 6[uD, u] + 3[u_x, D^2]$$

$$= 4[D^3, u] + 6[u, D^2]D + 6u[D, u] + 3[u_x, D^2]$$

$$= 4(u_{xxx} + \cancel{3u_{xx}D} + \cancel{3u_x D^2}) - 6(\cancel{u_{xx}} + \cancel{2u_x D})D$$

$$+ 6uu_x - 3(u_{xxx} + \cancel{2u_{xx}D})$$

$$= u_{xxx} + 6uu_x, \qquad (9.30)$$

which reproduces (9.24) as promised. Therefore

$$\boxed{\text{KdV for } u} \iff \boxed{L_t = [M, L]}$$

with

$$\boxed{\begin{aligned} L &= D^2 + u \\ M &= -(4D^3 + 6uD + 3u_x) \end{aligned}} \,. \qquad \text{LAX PAIR FOR KdV} \qquad (9.31)$$

• NOTE: Both L and M were differential operators (they involved $D = \frac{\partial}{\partial x}$), but $[L, M]$ is not: it just multiplies by a function, in this case $u_{xxx} + 6uu_x$.

For this reason $[L, M]$ is called "MULTIPLICATIVE".

This special property is required because (see (9.23)) it must be tha

$$[M(u), L(u)] = N(u),$$

where $N(u)$ is just a function of $u, u_x, u_{xx}, \ldots$, which appears in the PDE $u_t = N(u)$.

9.4 - Time evolution of the scattering data

- We have seen that if u evolves by the KdV eqn, then:

  1) the <u>EIGENVALUES</u> λ of $L(u)$ remain constant in t;

  2) the <u>EIGENFUNCTIONS</u> ψ evolve by $\boxed{\psi_t = M(u)\psi}$. (9.16)

$\boxed{Q}$ How does the <u>SCATTERING DATA</u> associated to $V=-u$ evolve in t?

$\boxed{A}$ Look at the $x \to \pm\infty$ <u>ASYMPTOTICS of (9.16)</u>.

- Recall that for KdV

$$\boxed{M(u) = -\left(4D^3 + 6uD + 3u_x\right)}. \tag{9.25}$$

Since $u, u_x \to 0$ as $x \to \pm\infty$ for all t according to our BC's, the <u>ASYMPTOTICS of $M(u)$</u>

$$\boxed{M(u) \to -4D^3} \quad \text{as } x \to \pm\infty \tag{9.32}$$

is <u>INDEPENDENT of u.</u>

- This is a <u>KEY POINT</u>: we can evolve the scattering data forward in t without knowing in advance what u evolves to!

You might worry about the normalization condition $\langle \psi_n, \psi_n \rangle = 1$ that we imposed to fix the normalization constants c_n of the bound states. Is it preserved under t-evolution? The answer is YES. This follows from $M(u)^\dagger = -M(u)$, which means $\langle \psi_1, M(u)\psi_2 \rangle = -\langle M(u)\psi_1, \psi_2 \rangle \quad \forall \psi_1, \psi_2$.
$\to$ * Ex... (and more soon...)

- Let's work out explicitly the t-evolution of the asymptotics of the scattering and bound state solutions.

1) CONTINUOUS SPECTRUM $(-\lambda = k^2 > 0)$:

Unnormalized SCATTERING SOLUTION (right-moving incident waves)

$$\psi_k(x;t) \approx \begin{cases} A(k;t)e^{ikx} + B(k;t)e^{-ikx} &, x \to -\infty \\ C(k;t)e^{ikx} &, x \to +\infty \end{cases} \tag{9.33}$$

Impose $\partial_t \psi_k(x;t) = M(u)\psi_k(x;t) \underset{x\to\pm\infty}{\approx} -4D^3\psi_k(x;t)$:

$$\begin{cases} A_t(k;t)e^{ikx} + B_t(k;t)e^{-ikx} = 4ik^3\left[A(k;t)e^{ikx} - B(k;t)e^{-ikx}\right] \\ C_t(k;t)e^{ikx} = 4ik^3 C(k;t)e^{ikx} \end{cases} \quad \begin{matrix} \forall k>0 \\ (\forall x \to \pm\infty) \end{matrix}$$

Equating coefficients of $e^{\pm ikx}$:

$$\Rightarrow \begin{cases} A_t(k;t) = 4ik^3 A(k;t) \\ B_t(k;t) = -4ik^3 B(k;t) \\ C_t(k;t) = 4ik^3 C(k;t) \end{cases} \tag{9.34}$$

$$\Rightarrow \begin{cases} A(k;t) = A(k;0)e^{4ik^3t} \\ B(k;t) = B(k;0)e^{-4ik^3t} \\ C(k;t) = C(k;0)e^{4ik^3t} \end{cases} \tag{9.35}$$

Divide ψ_k by $A(k;t)$ so that incoming waves have unit flux at all t:

$$\begin{aligned} \frac{B(k;t)}{A(k;t)} &= \boxed{R(k;t) = R(k;0)e^{-8ik^3t}} \\ \frac{C(k;t)}{A(k;t)} &= \boxed{T(k;t) = T(k;0)} \end{aligned} \tag{9.36}$$

• (9.33), (9.36) can be summarized in the <u>NORMALIZED SCATTERING SOLUTION</u> with asymptotics

$$\boxed{\psi_k(x) \approx \begin{cases} e^{ikx} + R(k;0)\, e^{-ik(x+8k^2t)} & , \; x \to -\infty \\ T(k)\, e^{ikx} & , \; x \to +\infty \end{cases}} \qquad (9.37)$$

The reflected time-dependent waves for ψ will eventually translate into <u>dispersive waves</u> for ψ moving to the left towards $x = -\infty$.

2) <u>DISCRETE SPECTRUM $(-\lambda = -\mu_n^2 < 0)$</u>

$$\psi_n(x;t) \approx \begin{cases} c_n(t)\, e^{\mu_n x} & , \quad x \to -\infty \\ d_n(t)\, e^{-\mu_n x} & , \quad x \to +\infty \end{cases} \qquad (9.38)$$

Impose $\quad \partial_t \psi_n(x;t) \underset{x \to \pm\infty}{\approx} -4D^3 \psi_n(x;t)$:

$$\begin{cases} \partial_t c_n(t) = -4\mu_n^3\, c_n(t) \\ \partial_t d_n(t) = +4\mu_n^3\, d_n(t) \end{cases} \qquad (9.39)$$

$$\Longrightarrow \quad \boxed{\begin{aligned} c_n(t) &= c_n(0)\, e^{-4\mu_n^3 t} \\ d_n(t) &= d_n(0)\, e^{+4\mu_n^3 t} \end{aligned}} \; . \qquad (9.40)$$

• (9.38), (9.40) can be summarized as

$$\boxed{\psi_n(x;t) \approx \begin{cases} c_n(0)\, e^{\mu_n(x-4\mu_n^2 t)} & , \; x \to -\infty \\ d_n(0)\, e^{-\mu_n(x-4\mu_n^2 t)} & , \; x \to +\infty \end{cases}} \; . \qquad (9.41)$$

These bound state solutions for ψ will eventually translate into solitons for u moving to the right at velocity $4\mu_n^2$.

- (9.36) and (9.40) describe the TIME EVOLUTION OF THE SCATTERING DATA, completing step (b) of the inverse scattering method.