Solitons III
(2021-22)

Exercises

October 1, 2021

0 Introduction

Ex 1 Numerical results seen in the lectures suggest that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{0.1}$$

has an exact solution of the form

$$u(x,t) = \frac{2}{\cosh^2(x - vt)} \tag{0.2}$$

for some constant velocity v. Verify this by direct substitution into the KdV equation and determine the value of v.

Ex 2

1. Show that if $u(x,t) = g(x,t)$ solves the KdV equation (0.1), then so does $u(x,t) = Ag(Bx, Ct)$, provided that the constants B and C are related to A in a specific way (which you should determine).
2. Apply this transformation to the basic KdV solution found in exercise 1 to construct a one-parameter family of one-soliton solutions of the KdV equation.
3. Find a formula relating the velocities to the heights for solitons in this one-parameter family. How does the width of a soliton in this family change if its velocity is rescaled by a factor of 4?

Ex 3 Show that if $u(x,t)$ solves the KdV equation and ϵ is a constant, then $v(x,t) := \frac{1}{\epsilon}u(x,t)$ solves the rescaled KdV equation

$$v_t + 6\epsilon vv_x + v_{xxx} = 0\,, \tag{0.3}$$

while $w(x,t) := \epsilon u(x, \epsilon t)$ solves the differently-rescaled KdV equation

$$w_t + 6ww_x + \epsilon w_{xxx} = 0\,. \tag{0.4}$$

Ex 4 Consider a pair of solitons with velocities m and n in the ball and box model, with $m > n$ and the faster soliton to the left of the slower one, with separation $l \geq n$ (*i.e.* there are $l \geq n$ empty boxes between the two solitons). Evolve various such initial conditions forward in time using the ball and box rule, for different values of m, n and l. Prove that the system always evolves into an oppositely-ordered pair of the same two solitons, and find a general formula for the phase shifts[1] of the solitons in terms of m and n.

[Optional:] What can go wrong if $l < n$? **[Hint:** Evolve the system backwards…]

Ex 5 In the two-colour (blue and red) ball and box model, we'll call a row of n consecutive balls a soliton if it keeps its form over time, so that after each time-step its only change is a possible (fixed) translation. There's no need for both colours to be represented, so a row of n blue balls, or a row of n red balls, is also a potential soliton. How many solitons of length n are there? What are their speeds?

Ex 6* The ball and box model can be further generalised to the M-colour ball and box model. The balls now come in M colours, 1, 2, …, M, and the time-evolution rule is generalised to say that first all balls of colour 1 are moved, then all of colour 2, and so on, with a single time-step being completed once all balls of all colours have been moved. How many solitons of length n are there in this model? Again, there is no need for every colour to be present in a given soliton. You might start by classifying the 'top-speed' solitons of length n, that is, those that move at speed n.

Ex 7* Investigate the scattering of solitons in the two-colour ball and box model. You should find that the lengths of top-speed solitons are preserved under collisions, but their forms can change. Try to formulate a general rule for this behaviour. Can you generalise it to the M-colour model?

[1] The phase shift of a soliton is defined to be the shift of its position, at a time in the far future, relative to the position it would have had at the same time if the other soliton hadn't been there.

1 Waves, dispersion and dissipation

Ex 8

1. Express d'Alembert's general solution of the wave equation $u_{tt} - u_{xx} = 0$ in terms of the initial conditions $u(x,0) = p(x)$ and $u_t(x,0) = q(x)$.
2. Find a relation between $p(x)$ and $q(x)$ which produces a single wave travelling to the right.

Ex 9 The wave profile

$$\phi(x,t) = \cos(k_1 x - \omega(k_1)t) + \cos(k_2 x - \omega(k_2)t) \tag{1.1}$$

is a superposition of two plane waves. Rewrite ϕ as a product of cosines, and use this to sketch the wave profile when $|k_1 - k_2| \ll |k_1|$. Find the velocity at which the envelope of the wave profile moves (the **group velocity**), again for $k_1 \approx k_2$; in the limit $k_1 \to k_2$ verify that this reduces to $d\omega/dk$, consistent with the result obtained in lectures.

Ex 10

1. Completing the square, derive the formula
$$\int_{-\infty}^{+\infty} dk\ e^{-A(k-\bar{k})^2} e^{ikB} = \sqrt{\frac{\pi}{A}}\ e^{i\bar{k}B} e^{-B^2/(4A)}\ . \tag{1.2}$$
(You can quote the result $\int_{-\infty}^{+\infty} dk\ e^{-Ak^2} = \sqrt{\pi/A}$ for $A > 0$.)
2. For the Gaussian wavepacket (where Re denotes the real part)
$$u(x,t) = \mathrm{Re} \int_{-\infty}^{+\infty} dk\ e^{-a^2(k-\bar{k})^2} e^{i(kx-\omega(k)t)}\ , \tag{1.3}$$
expand $\omega(k)$ to second order in $k - \bar{k}$, and then use the result of part 1 to derive a better approximation for $u(x,t)$ than that obtained in lectures.
3. Given that a function of the form $e^{-(x-x_0)^2/C}$ describes a profile centred at x_0 with width^{-2} equal to the real part of C^{-1}, show that the result of part 2 is a wave profile moving at velocity $\omega'(\bar{k})$, with width^2 increasing with time as $4a^2 + \omega''(\bar{k})^2 t^2/a^2$. (Hence, for $\omega'' \neq 0$, the wave disperses.)

Ex 11 Find the dispersion relation and the phase and group velocities for:

(a) $u_t + u_x + \alpha u_{xxx} = 0\,;$

(b) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}$.

Ex 12 For which values of n does the equation

$$u_t + u_x + u_{xxx} + \frac{\partial^n u}{\partial x^n} = 0 \tag{1.4}$$

admit “physical" dissipation? (A wave is said to have physical dissipation if the amplitude of plane waves decreases with time.)

2 Travelling waves

Ex 13 Find (if possible) real non-singular travelling wave solutions of the following equations, satisfying the given boundary conditions:

1. Modified KdV (mKdV) equation:
$$\begin{aligned} &u_t + 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\ . \end{aligned} \tag{2.1}$$
2. 'Wrong sign' mKdV equation:
$$\begin{aligned} &u_t - 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\ . \end{aligned} \tag{2.2}$$
3. ϕ^4 theory:
$$\begin{aligned} &u_{tt} - u_{xx} + 2u(u^2 - 1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to -1 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to +1 \ \text{ as } x \to +\infty\ . \end{aligned} \tag{2.3}$$
4. ϕ^6 theory:
$$\begin{aligned} &u_{tt} - u_{xx} + u(u^2 - 1)(3u^2 - 1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to 0 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to 1 \ \text{ as } x \to +\infty\ . \end{aligned} \tag{2.4}$$
5. Burgers equation:
$$\begin{aligned} &u_t + uu_x - u_{xx} = 0 \\ &u \to u_0,\ u_x \to 0 \ \text{ as } x \to -\infty \\ &u \to u_1,\ u_x \to 0 \ \text{ as } x \to +\infty\ , \end{aligned} \tag{2.5}$$
   where u_0 and u_1 are real constants with $u_0 > u_1 > 0$.
   [Hint: Start by showing that the boundary conditions relate the velocity v of the travelling wave to the sum of the constants u_0 and u_1.]
6. * Generalised KdV equation with $n = 1, 2, 3, \ldots$:
$$\begin{aligned} &u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\ . \end{aligned} \tag{2.6}$$

Ex 14* Using the analogy with the classical mechanics of a point particle moving in one spatial dimension, determine the qualitative behaviour of travelling wave solutions of the KdV equation on a circle, for which the integration constants A and B are non-zero.

3 Topological lumps and the Bogomol'nyi bound

Ex 15 This exercise involves the infinite chain of identical coupled pendula of section 2.3, whose equations of motion reduce to the sine-Gordon equation in the continuum limit $a \to 0$. We will simplify expression by setting $g = L = \frac{M}{a} = 1$. Let $\theta_n(t)$ be the angle to the vertical of the n-th pendulum ($n \in \mathbb{Z}$), which is hung at the position $x = na$ along the chain, at time t. The configuration of the system at time t is then specified by the collection of angles $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

1. Starting from the force (note: m is a dummy variable)

$$F_n(\{\theta_m\}) = -a\sin\theta_n + \frac{1}{a}(\theta_{n+1} - \theta_n) + \frac{1}{a}(\theta_{n-1} - \theta_n) \tag{3.1}$$

acting on the n-th pendulum, deduce the potential energy

$$V(\{\theta_m\}) = \sum_{n=-\infty}^{+\infty} (\cdots) \tag{3.2}$$

such that $F_n = -\frac{\partial V}{\partial \theta_n}$ for all $n \in \mathbb{Z}$, and fix the integration constant by requiring that the potential energy be zero when all pendula point down: $V(\{0\}) = 0$.

2. Show that in the continuum limit $a \to 0$, the potential energy computed above becomes

$$V = \int_{-\infty}^{+\infty} dx \, \left[(1 - \cos\theta) + \frac{1}{2}\theta_x^2 \right] \, , \tag{3.3}$$

and the kinetic energy

$$T(\{\theta_m\}) = \frac{a}{2} \sum_{n=-\infty}^{+\infty} \dot{\theta}_n^2 \tag{3.4}$$

becomes

$$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2}\theta_t^2 \, , \tag{3.5}$$

where the function $\theta(x, t)$ is the continuum limit of $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

[**Hint**: in the continuum limit, $a \sum\limits_{n=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty} dx$.]

Ex 16 A field $u(x,t)$ has kinetic energy T and potential energy V, where

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \, \frac{1}{2} u_t^2 \, , \\ V &= \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] \, , \end{aligned} \tag{3.6}$$

and a and $\lambda > 0$ are (real) constants. (This is a version of the 'ϕ^4' theory. It's called like that because the scalar potential is quartic, and the field u is usually called ϕ.) The equation of motion for u is

$$u_{tt} - u_{xx} + 2\lambda u (u^2 - a^2) = 0 \, . \tag{3.7}$$

1. If u is to have finite energy, what boundary conditions must be imposed on u, u_x and u_t at $x = \pm\infty$?
2. Find the general travelling-wave solution(s) to the equation of motion, consistent with the boundary conditions found in part 1. Compute the total energy $E = T+V$ for these solutions. For which velocity do the solutions have the lowest energy?
3. One of the possible boundary conditions for part 1 implies that u is a kink, with $[u(x)]_{x=-\infty}^{x=+\infty} = 2a$. Use the Bogomol'nyi argument to show that the total energy $E = T+V$ of that configuration is bounded from below by $C\sqrt{\lambda}a^3$, where C is a constant that you should determine, and find the solution u which saturates this bound. Verify that this solution agrees with the lowest-energy solution of part 2.

Ex 17

1. Explain why the Bogomol'nyi argument given in the lectures fails to provide a useful bound on the energy of a two-kink solution of the sine-Gordon equation (a two-kink solution is one with topological charge $n - m$ equal to 2). What is the most that can be said about the energy of a k-kink?
2. For a sine-Gordon field u, generalise the Bogomol'nyi argument to show that

$$\int_A^B dx \, \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + (1 - \cos u) \right] \geq \pm 4 \left[\cos \frac{u}{2} \right]_A^B \, . \tag{3.8}$$

3. * Use this result and the intermediate value theorem (look it up if necessary!) to show that if the field u has the boundary conditions of a k-kink, then its energy is at least k times that of a single kink. Can this bound be saturated?

Ex 18 A system on the finite interval $-\pi/2 \leq x \leq \pi/2$ is defined by the following expressions for the kinetic energy T and the potential energy V:

$$\begin{aligned} T &= \int_{-\pi/2}^{\pi/2} dx \ \frac{1}{2} u_t^2 \\ V &= \int_{-\pi/2}^{\pi/2} dx \ \frac{1}{2} \left(u_x^2 + 1 - u^2 \right) \ . \end{aligned} \tag{3.9}$$

The function $u(x,t)$ satisfies the boundary condition $|u(\pm\pi/2, t)| = 1$ and is required to satisfy $|u(x,t)| \leq 1$ everywhere. Show that with "kink" boundary conditions, the total energy E is bounded below by a positive constant, and find a solution for which the bound is saturated.

4 Conservation laws

Ex 19 Check explicitly that the energy

$$E = \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \mathbb{V}(u) \right] \tag{4.1}$$

and the momentum

$$P = - \int_{-\infty}^{+\infty} dx \; u_t u_x \tag{4.2}$$

of a relativistic field $u(x,t)$ in 1 space and 1 time dimensions are conserved when the equation of motion

$$u_{tt} - u_{xx} = -\mathbb{V}'(u) \tag{4.3}$$

and the boundary conditions

$$u_t, \; u_x, \; \mathbb{V}(u), \; \mathbb{V}'(u) \underset{x \to \pm\infty}{\longrightarrow} 0 \qquad \forall t \tag{4.4}$$

are satisfied.

Ex 20

1. Compute the conserved topological charge, energy and momentum of a sine-Gordon kink moving with velocity v, and check that the results do not depend on time. [**Hint**: The integral (A.9) might be useful. For the scalar potential term in the energy, write $1 - \cos(u) = 2 \sin^2(u/2)$, plug in the kink solution and manipulate the result using trigonometric formulae until (A.9) becomes useful.]
   Confirm that for $|v| \ll 1$ the energy and the momentum take the forms
   $$E = M + \frac{1}{2} M v^2 + \mathcal{O}(v^4) \; , \qquad P = Mv + \mathcal{O}(v^3) \tag{4.5}$$
   where the 'mass' M is the energy of the static kink, which appears in the Bogomol'nyi bound.
2. * If you are fearless and have time on your hands, try also to compute the conserved spin 3 charge
   $$Q_3 = \int_{-\infty}^{+\infty} dx \, \left[u_{++}^2 - \frac{1}{4} u_+^4 + u_+^2 \cos u \right] \tag{4.6}$$
   for the sine-Gordon kink. The integrals are not at all straightforward, but can be evaluated using appropriate changes of variables. (Did I write fearless?)

Ex 21 Find three conserved charges for the mKdV equation (2.1) of Ex 13.1, which involve u, u^2 and u^4 respectively. The boundary conditions on $u(x,t)$ are u, u_x and $u_{xx} \to 0$ as $|x| \to \infty$. Evaluate these quantities for the travelling-wave solution found in Ex 13.1. The list of definite integrals at the end of the problems sheet might help.

Ex 22 Show that u is a conserved density for Burgers' equation (2.5). Why is this result of no use in analysing the travelling wave solution of Ex 13.5?

Ex 23 Consider the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$.

1. Show that $\rho_1 \equiv u$, $\rho_2 \equiv u^2$ and $\rho_* \equiv xu - 3tu^2$ are all conserved densities, so that

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_* = \int_{-\infty}^{+\infty} dx\ (xu - 3tu^2) \tag{4.7}$$

   are all conserved charges.

2. Evaluate the conserved charges Q_1, Q_2 and Q_* for the one-soliton solution centred at x_0 and moving with velocity $v = 4\mu^2$:

$$u_{\mu,\,x_0}(x,t) = 2\mu^2 \mathrm{sech}^2 \left[\mu(x - x_0 - 4\mu^2 t)\right]\ . \tag{4.8}$$

3. According to the KdV equation, the initial condition $u(x,0) = 6\,\mathrm{sech}^2(x)$ is known to evolve into the sum of two well-separated solitons with different velocities $v_1 = 4\mu_1^2$ and $v_2 = 4\mu_2^2$ at late times. Use the conservation of Q_1 and Q_2 to determine v_1 and v_2.

4. A two-soliton solution separates as $t \to -\infty$ into two one-solitons u_{μ_1,x_1} and u_{μ_2,x_2}. As $t \to +\infty$, two one-solitons are again found, with μ_1 and μ_2 unchanged but with x_1, x_2 replaced by y_1, y_2. Use the conservation of Q_* to find a formula relating the *phase shifts* $y_1 - x_1$ and $y_2 - x_2$ of the two solitons.

Ex 24

1. Show that if $u(x,t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where λ is a constant and $v(x,t)$ some other function, then v satisfies

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t + 6\lambda v_x - 6v^2 v_x + v_{xxx}\right) = 0\,.$$

2. Compute the Gardner transform expansion

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t)\varepsilon^n \tag{4.9}$$

up to order ε^4. Use the results to find the conserved charges $\widetilde{Q}_3$ and $\widetilde{Q}_4$, where

$$\widetilde{Q}_n = \int_{-\infty}^{+\infty} dx\ w_n\ . \tag{4.10}$$

Show that $\widetilde{Q}_3$ is the integral of a total x-derivative (and hence is zero), while $\widetilde{Q}_4 = \alpha\, Q_3$, where

$$Q_3 = \int_{-\infty}^{+\infty} dx\ \left(u^3 - \frac{1}{2}u_x^2\right) \tag{4.11}$$

is the third KdV conserved charge (the 'energy') and α a constant that you should determine. * If you're feeling energetic, try to compute $\widetilde{Q}_5$ and $\widetilde{Q}_6$ as well.

Ex 25 This question is also about the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

1. Evaluate the first three KdV conserved charges

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_3 = \int_{-\infty}^{+\infty} dx\ \left(u^3 - \frac{1}{2}u_x^2\right) \tag{4.12}$$

for the initial state $u(x,0) = A\,\mathrm{sech}^2(Bx)$, where A and B are constants.

2. The initial state

$$u(x,0) = N(N+1)\,\mathrm{sech}^2(x)\ , \tag{4.13}$$

where N is an integer, is known to evolve at late times into N well-separated solitons, with velocities $4k^2$, $k = 1 \ldots N$. So for $t \to +\infty$, this solution approaches the sum of N single well-separated solitons

$$u(x,t) \approx \sum_{k=1}^{N} 2\mu_k^2\,\mathrm{sech}^2\big[\mu_k(x - x_k - 4\mu_k^2 t)\big]\ , \tag{4.14}$$

where $\mu_1, \ldots, \mu_N$ are N different constants. Since Q_1, Q_2 and Q_3 are conserved, their values at $t = 0$ and $t \to +\infty$ must be equal. Use this fact to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers.

3. * Use Q_4 and Q_5 and the method just described to find the sum of the first N seventh and ninth powers, $\sum_{k=1}^{N} k^7$ and $\sum_{k=1}^{N} k^9$.

5 The Bäcklund transform

Ex 26 1. Show that the pair of equations

$$\begin{aligned}(u-v)_+ &= \sqrt{2}\, e^{(u+v)/2} \\ (u+v)_- &= \sqrt{2}\, e^{(u-v)/2}\end{aligned} \tag{5.1}$$

provides a Bäcklund transformation linking solutions of $v_{+-}=0$ (the wave equation in light-cone coordinates) to those of $u_{+-}=e^u$ (the Liouville equation).

2. Starting from d'Alembert's general solution $v=f(x^+)+g(x^-)$ of the wave equation, use the Bäcklund transform (5.1) to obtain the corresponding solutions of the Liouville equation for u. [**Hint**: Set $u(x^+,x^-)=2U(x^+,x^-)+f(x^+)-g(x^-)$. You might simplify the notation by setting $f(x^+)=\log(F'(x^+))$ and $g(x^-)=-\log(G'(x^-))$, where prime means first derivative.]

Ex 27 Consider the Bäcklund transform

$$v_x+\frac{1}{2}uv=0 \tag{5.2}$$

$$v_t+\frac{1}{2}u_xv-\frac{1}{4}u^2v=0\,. \tag{5.3}$$

1. Show that (5.2) and (5.3) together imply that v satisfies the linear heat equation $v_t=v_{xx}$, while u satisfies Burgers' equation $u_t+uu_x-u_{xx}=0$.
[**Hint**: for the first, solve (5.2) for u and substitute in (5.3); for the second, start by cross-differentiating.]

2. Find the *general* travelling-wave solution for $v(x,t)$ and, via the Bäcklund transform, re-obtain the travelling-wave for Burgers' equation found in question (2.5).

3. * The linear equation satisfied by $v(x,t)$ allows for the linear superposition of solutions. Use this fact, and your answers to part 2, to construct solutions for v and then u which describe the interaction of *two* travelling waves.

4. * Sketch your solutions functions of x at fixed times both before and after the interaction, and also draw their trajectories in the (x,t) plane, perhaps starting with the help of a computer. Are the travelling waves of Burgers' equation true solitons, in the sense given in lectures?
[**Hints**: Examine the asymptotics of the solution viewed from frames moving at various velocities V (that is, set $X_V=x-Vt$ and consider $t\to\pm\infty$ keeping X_V finite). This should allow you to isolate various travelling waves in these limits, and to decide whether they preserve their form under interactions. For definiteness, consider the case $c_1>c_2>0$, where c_1 and c_2 are the velocities of the two separate travelling waves before they were superimposed. A further hint: as well as the 'expected' special values for V, namely c_1 and c_2, be careful about what happens when $V=c_1+c_2$.]

Ex 28 1. Show that the two equations

$$\begin{aligned} v_x &= -u - v^2 \\ v_t &= 2u^2 + 2uv^2 + u_{xx} - 2u_x v \end{aligned} \tag{5.4}$$

are a Bäcklund transform relating solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5.5}$$

and the wrong sign modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0\,. \tag{5.6}$$

(Note the appearance of the Miura transform in (5.4).)

2. Taking $u = c^2$, where c is a constant, as a seed solution of the KdV equation, find the corresponding solution of the wrong sign mKdV equation.

Ex 29 The 2-soliton solution of the sine-Gordon equation with Bäcklund parameters a_1 and a_2 is

$$u(x,t) = 4 \arctan\left(\mu \, \frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}}\right), \qquad \theta_i = \varepsilon_i \gamma_i (x - v_i t - \bar{x}_i) \tag{5.7}$$

where $\mu = (a_2{+}a_1)/(a_2{-}a_1)$, $v_i = (a_i^2{-}1)/(a_i^2{+}1)$, $\gamma_i = 1/\sqrt{1{-}v_i^2}$, $\varepsilon_i = \text{sign}(a_i)$, and $\bar{x}_1$ and $\bar{x}_2$ are constants, as in the lectures. Rewriting u as a function of $X_V \equiv x - Vt$ and t, show that, for $V \neq v_1, v_2$ (and $v_1 \neq v_2$)

$$\lim_{\substack{t\to\infty \\ X_V \ \text{finite}}} u = 2n\pi \ ,$$

where n is an integer. If $v_2 > v_1 > 0$ and $\varepsilon_i = 1$, how does the parity of n (whether it is even or odd) depend on the value of v relative to v_1 and v_2?
[**Hints**: First show that $|\theta_i| \to +\infty$ as $t \to \pm\infty$; then consider each of the four possible options $(\theta_1, \theta_2) \to (+\infty, +\infty)$, $(-\infty, -\infty)$, $(+\infty, -\infty)$, $(-\infty, +\infty)$. Remember that $\arctan(0) = m\pi$ and $\arctan(\pm\infty) = \pm\pi/2 + m\pi$, where the ambiguities of $m\pi$, $m \in \mathbb{Z}$, encode the multivalued nature of the arctan function.]

Ex 30 Find the asymptotics of the 2-soliton sine-Gordon solution defined in equation (5.7), in the case $a_2 > a_1 > 0$, as $t \to \pm\infty$ with $X_{v_2} \equiv x{-}v_2 t$ held finite.

Ex 31 Show by direct analysis (as in the lectures) that taking a_1 and a_2 of opposite signs in (5.7) results in a two-kink, or two-antikink, solution to the sine-Gordon equation.

Ex 32

1. The argument of the arctangent in the sine-Gordon 2-soliton solution (5.7) is a continuous function of x for all $x \in \mathbb{R}$. Show that, in particular, it is never infinite. What does this imply about the range of u? [**Hint**: consider the graph of $\tan u/4$.]
2. By taking the limits of this function as $x \to \pm\infty$ (with $t = \bar{x}_1 = \bar{x}_2 = 0$ for simplicity), show that the topological charge of the two-soliton solution (5.7) is 0 if $\mathrm{sign}(a_1) = \mathrm{sign}(a_2)$, and ± 2 if $\mathrm{sign}(a_1) = -\mathrm{sign}(a_2)$, in units where the topological charge of a kink is 1.

Ex 33 Consider the two-soliton solution of the sine-Gordon equation (5.7) with complex Bäcklund parameters $a_1 = \overline{a_2} := a \in \mathbb{C}$ and with vanishing integration constants, as is appropriate to find the breather solution. Show that

$$\begin{aligned} \mathrm{Re}(\theta_1) &= +\mathrm{Re}(\theta_2) = \gamma(x - vt)\cos\varphi \ , \\ \mathrm{Im}(\theta_1) &= -\mathrm{Im}(\theta_2) = \gamma(vx - t)\sin\varphi \ , \end{aligned} \tag{5.8}$$

where $\varphi = \arg(a)$ and

$$\begin{aligned} v &= \frac{|a|^2 - 1}{|a|^2 + 1} \\ \gamma &= \frac{1}{\sqrt{1 - v^2}} = \frac{1 + |a|^2}{2|a|} \ . \end{aligned} \tag{5.9}$$

Ex 34 The *stationary* breather solution of the sine-Gordon equation (that is the breather solution with $v = 0$) has the form

$$\tan\frac{u}{4} = \frac{\cos\varphi}{\sin\varphi} \cdot \frac{\sin(t\sin\varphi)}{\cosh(x\cos\varphi)} \ . \tag{5.10}$$

Show that in the limit $\varphi \to 0$, in which the kink and antikink that form the breather are very loosely bound, the time period τ of a single oscillation of the breather scales like $\tau \sim |\varphi|^{-1}$, and the spatial size $x_{\max}$ of the breather scales like $x_{\max} \sim -\log\varphi$.
[**Hint**: You could define $x_{\max}$ as the value of x at which $\tan(u/4) = 1$ when the oscillatory factor in the numerator is at its maximum. Focus only on the parametric dependence on φ, ignoring all numerical factors.]

6 The Hirota method

Ex 35 We have seen in lectures that the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$ that describes the profile of a wave translates into the following equation for the new variable $w(x,t) = \int dx\ u$:

$$w_t + 3w_x^2 + w_{xxx} = 0 \ . \tag{6.1}$$

Let $w = 2\frac{\partial}{\partial x}\log f = 2\frac{f_x}{f}$ where $f(x,t)$ is a nowhere vanishing function of x and t, so that $u = 2\frac{\partial^2}{\partial x^2}\log f$. The aim of this exercise is to rewrite (6.1) as an equation for f.

1. Express w_t, w_x, w_{xx} and w_{xxx} in terms of f and its derivatives.
2. Show that the equation (6.1) can be rewritten as

$$f f_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + f f_{xxxx} = 0 \ , \tag{6.2}$$

which is known as the *quadratic form* of the KdV equation.

Ex 36 The Hirota bilinear differential operator $D_t^m D_x^n$ is defined for any pair of natural numbers (m,n) by

$$D_t^m D_x^n(f,g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x\\ t'=t}} \tag{6.3}$$

and maps a pair of functions $(f(x,t), g(x,t))$ into a single function.

1. Prove that the Hirota operators $B_{m,n} := D_t^m D_x^n$ are bilinear, *i.e.* for all constants a_1, a_2

$$\begin{aligned} B_{m,n}(a_1 f_1 + a_2 f_2, g) &= a_1 B_{m,n}(f_1,g) + a_2 B_{m,n}(f_2,g) \ , \\ B_{m,n}(f, a_1 g_1 + a_2 g_2) &= a_1 B_{m,n}(f,g_1) + a_2 B_{m,n}(f,g_2) \ . \end{aligned} \tag{6.4}$$

2. Prove the symmetry property

$$B_{m,n}(f,g) = (-1)^{m+n} B_{m,n}(g,f) \ . \tag{6.5}$$

3. Compute the Hirota derivatives $D_t^2(f,g)$ and $D_x^4(f,g)$, and verify that your expression for the latter is consistent with the result for $D_x^4(f,f)$ given in lectures.

Ex 37 Define a “non-Hirota" bilinear differential operator $\tilde{D}_t^m \tilde{D}_x^n$ by

$$\tilde{D}_t^m \tilde{D}_x^n(f,g) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x\\ t'=t}} \tag{6.6}$$

(note the plus signs!).

1. Compute $\tilde{D}_x(f,g)$ and $\tilde{D}_t(f,g)$, verifying that in both cases the answer is given by the corresponding ‘ordinary’ derivative of the product $f(x,t)g(x,t)$.
2. How does this result generalise for arbitrary non-Hirota differential operators (6.6)? Prove your claim.
3. Compare your answer with the Hirota operators defined above.

Ex 38

1. If $\theta_i = a_i x + b_i t + c_i$, prove that

$$D_t D_x(e^{\theta_1}, e^{\theta_2}) = (b_1 - b_2)(a_1 - a_2)e^{\theta_1+\theta_2}.$$

2. Prove the corresponding result for $D_t^m D_x^n(e^{\theta_1}, e^{\theta_2})$, as quoted in lectures.

Ex 39 Prove that

$$D_t^m D_x^n(f,1) = \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n} f\ . \tag{6.7}$$

Ex 40 Consider the function f, such that $u = 2\frac{\partial^2}{\partial x^2}\log f$ is the KdV field, which corresponds to a 2-soliton solution:

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 = 1 + \epsilon\left(e^{\theta_1} + e^{\theta_2}\right) + \epsilon^2\left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1+\theta_2}\ , \tag{6.8}$$

where $\theta_i = a_i x - a_i^3 t + c_i$, with a_i and c_i constants. Check that $B(f_1, f_2) = 0$ and $B(f_2, f_2) = 0$, where $B = D_x(D_t + D_x^3)$, and show that this implies that the expansion (6.8), which is truncated at order ϵ^2, is a solution of the bilinear form of the KdV equation.

Ex 41* Derive the solution of the bilinear form of the KdV equation $D_x(D_t + D_x^3)(f, f) = 0$ which represents the 3-soliton solution, in the form

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \tag{6.9}$$

where $f_1 = \sum_{i=1}^{3} e^{\theta_i}$. [This includes proving that the higher order terms in the ϵ expansion can be consistently set to zero, as in Ex 40.]

Ex 42 Show that the Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0 \tag{6.10}$$

can be written in the bilinear form

$$(D_t^2 - D_x^2 - D_x^4)(f, f) = 0 \tag{6.11}$$

where $u = 2\frac{\partial^2}{\partial x^2} \log f$.

Ex 43 Show that the following higher-dimensional version of the KdV equation,

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{6.12}$$

for the field $u(x, y, t)$, also known as the Kadomtsev-Petviashvili (KP) equation, can be written in the bilinear form

$$(D_t D_x + D_x^4 + 3\sigma^2 D_y^2)(f, f) = 0 \tag{6.13}$$

where $u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log f(x, y, t)$.

7 Exam-style problem

Ex 44 The complex field $u(x,t)$ obeys the equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0 \ , \tag{7.1}$$

where $i = \sqrt{-1}$, and the boundary conditions

$$u, u_x, u_{xx} \to 0 \qquad \text{as} \quad x \to \pm\infty \ . \tag{7.2}$$

1. Show that the quantities

$$\begin{aligned} Q_1 &= \int_{-\infty}^{+\infty} dx \ |u|^2 \\ Q_2 &= \int_{-\infty}^{+\infty} dx \ \mathrm{Im}(\bar{u}u_x) \\ Q_3 &= \int_{-\infty}^{+\infty} dx \ \left(\frac{1}{2}|u_x|^2 + C|u|^4\right) \end{aligned} \tag{7.3}$$

   are conserved provided that the constant C takes a value that you should find. (Here Im denotes the imaginary part and a bar denotes complex conjugation.)

2. Show that given a 'seed' solution $u(x,t)$ of equation (7.1),

$$u^{(v)}(x,t) := u(x - vt, t)e^{i(Ax+Bt)} \tag{7.4}$$

   is also a solution for all $v \in \mathbb{R}$, provided that the constants A and B depend on v in a way that you should find.

3. Determine the functional dependence of the conserved charges Q_1, Q_2, Q_3 in (7.3) on the parameter v that labels the one-parameter family of solution (7.4).

4. Find all solutions of the form

$$u(x,t) = \rho(x)e^{i\varphi(t)} \tag{7.5}$$

   of equation (7.1) with boundary conditions (7.2), where ρ are φ are real and $u(x,0)$ is a real even function of x. [You can use the integrals at the end of the problem sheet.] Apply the method of part 2 to this seed solution to find the associated one-parameter family of solutions $u^{(v)}(x,t)$.

A Useful integrals

You can freely quote the following formulae, athough deriving them may be instructive:

- **Indefinite integrals:** [**Note:** the integration constant is in principle complex]

$$\int \frac{dx}{x\sqrt{1-x}} = -2\text{arcsech}(\sqrt{x}) \tag{A.1}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\text{arcsech}(x) \tag{A.2}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\text{arccosech}(x) \tag{A.3}$$

$$\int \frac{dx}{\sin(x/2)} = 2\log\tan(x/4) \tag{A.4}$$

$$\int \frac{dx}{\cosh(x)} = 2\arctan(e^x) \tag{A.5}$$

$$\int \frac{dx}{1-x^2} = \text{arctanh}(x) \tag{A.6}$$

$$\int dx\ \sqrt{1-x^2} = \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin(x)\right] \tag{A.7}$$

$$\int \frac{dx}{\cos^2(x)} = \tan(x) \tag{A.8}$$

$$\int \frac{dx}{\cosh^2(x)} = \tanh(x) \tag{A.9}$$

- **Definite integrals:**

$$\int_{-\infty}^{+\infty} dx\ e^{-Ax^2} = \sqrt{\frac{\pi}{A}} \qquad (A > 0) \tag{A.10}$$

$$\int_{-\infty}^{+\infty} dx\ \text{sech}^{2n}(x) = \frac{2^{2n-1}((n-1)!)^2}{(2n-1)!} \tag{A.11}$$

Note: the result of the Gaussian integral (A.10) does not change if the integration variable x is shifted by a finite imaginary amount c, namely if you replace $x \to x + ic$.