LORENTZ TRANSFORMATIONS & LIGHTCONE COORDINATES

The equation of motion (EoM)

$$u_{tt} - u_{xx} = -V'(u) \tag{1}$$

of a relativistic field theory in 1+1 (1 space + 1 time) dimensions is invariant under

$$\begin{aligned} t &\longmapsto t' = t + a \qquad \text{TIME TRANSLATIONS} \\ x &\longmapsto x' = x + b \qquad \text{SPACE TRANSLATIONS} \end{aligned} \tag{2}$$

for any $a, b \in \mathbb{R}$, since $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$.

Therefore if $u(x,t)$ is a solution of (1), $u(x+b, t+a)$ is also a solution, for all $a, b \in \mathbb{R}$.

(By Noether's theorem, this in turn implies that energy and momentum are conserved.)

In addition to these 'obvious' symmetries, relativistic field equations like (1) have another more interesting group of symmetries, which rely on the particular combination of derivatives $(\partial_t^2 - \partial_x^2)$ appearing in (1): they are invariant under

$$\begin{cases} x \longmapsto x' = \gamma(x - vt) \\ t \longmapsto t' = \gamma(t - vx) \end{cases} \qquad \text{LORENTZ TRANSFORMATIONS (or LORENTZ BOOSTS)} \tag{3}$$

where $v \in (-1, 1)$ and $\gamma = \frac{1}{\sqrt{1-v^2}}$. [The speed of light is set to 1.]

This corresponds to switching to the point of view ("REFERENCE FRAME") of someone moving at velocity v in the original coordinates (x,t).

* Ex: Use the chain rule to show that $\partial_{t'}^2 - \partial_{x'}^2 = \partial_t^2 - \partial_x^2$, hence (1) is invariant under (3).

Therefore if $u(x,t)$ is a solution of (1), $u(\gamma(x-vt), \gamma(t-vx))$ is also a solution.
In particular, consider a static solution $u = u_0(x)$, which satisfies $\frac{d^2}{dx^2} u_0(x) = \left.\frac{dV}{du}\right|_{u=u_0(x)}$.
We can "boost" it using (3) to get a solution of (1) moving at velocity v:

$$u = u_v(x,t) = u_0(\gamma(x - vt)).$$

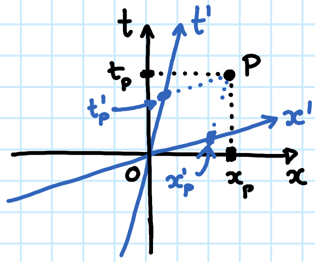
This explains the relation between a static kink and a kink moving at velocity v.

It is customary to set

$$\gamma = \cosh\vartheta \equiv c\,, \qquad \gamma v = \sinh\vartheta \equiv s \qquad (\text{Note: } \gamma^2 - \gamma^2 v^2 = 1) \tag{4}$$

and write the transformation (3) as

$$\begin{pmatrix} x \\ t \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{5}$$



using a 2x2 matrix of unit determinant.

The real parameter ϑ is called the RAPIDITY of the boost.

~ o ~

The set of all Lorentz boosts is a "group". Composing a boost of rapidity θ_1 with a boost of rapidity θ_2 gives another boost (of rapidity $\theta_1 + \theta_2$):

$$\begin{pmatrix} \cosh\theta_2 & -\sinh\theta_2 \\ -\sinh\theta_2 & \cosh\theta_2 \end{pmatrix} \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 \\ -\sinh\theta_1 & \cosh\theta_1 \end{pmatrix} = \begin{pmatrix} \cosh(\theta_1+\theta_2) & -\sinh(\theta_1+\theta_2) \\ -\sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{pmatrix}$$

This is called the LORENTZ GROUP in 1+1 dimensions, and it is "SO(1,1)", the group of 2x2 real matrices S s.t.

$$\det S = 1 \qquad \text{and} \qquad S^T \eta S = \eta\,, \quad \text{where} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

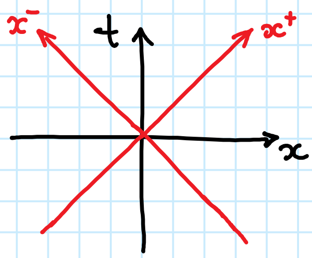
~ o ~

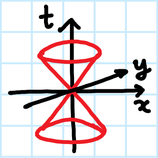
It is convenient, especially in 1+1 dimensions, to switch to

$$\text{LIGHT-CONE COORDINATES} \qquad x^{\pm} = \tfrac{1}{2}(t \pm x) \tag{6}$$

$$\iff \quad t = x^+ + x^-\,, \quad x = x^+ - x^-\,,$$

so called because the set of light rays through a point (which have x^+ or $x^- = $ const.) is a "cone":



(It looks more like a cone in higher dimension: )

The main advantage of light-cone coordinates is that they diagonalize the Lorentz boosts (5):

$$x^{\pm} = \frac{1}{2}(t \pm x) \quad \longmapsto \quad (x')^{\pm} = \frac{1}{2}(t' \pm x') = \frac{1}{2}\left[(ct - sx) \pm (cx - st)\right] \tag{7}$$
$$= (c \mp s) \cdot \frac{1}{2}(t \pm x) \quad = e^{\mp\theta} \cdot x^{\pm} \, .$$

So the light-cone coordinates are rescaled by $\exp(\mp \text{rapidity})$ under a Lorentz boost, and the partial derivatives with respect to them are rescaled by the inverse exponentials:

$$\partial_{\pm} \equiv \frac{\partial}{\partial x^{\pm}} \quad \longmapsto \quad \partial'_{\pm} \equiv \frac{\partial}{\partial (x')^{\pm}} = e^{\pm\theta} \cdot \partial_{\pm} \tag{8}$$

As we saw in the lecture, $\partial_t^2 - \partial_x^2 = \partial_+ \partial_-$, and this is invariant under Lorentz boosts, because (θ is a constant parameter)

$$\partial_+ \partial_- \longmapsto e^{\theta} \partial_+ \, e^{-\theta} \partial_- = \partial_+ \partial_- \, . \tag{9}$$

This shows that the EoM (1) is invariant under Lorentz transformations.

<u>NOTE</u>: an object U_s of SPIN s transforms as follows under Lorentz boosts:

$$U_s \longmapsto U'_s = e^{s\theta} \cdot U_s \, , \tag{10}$$

so objects of different spin transform differently.