PROBLEMS CLASS 6

16/2/2024

Ex 58 Let $D = \frac{d}{dx}$ and let $g(x)$ be a general function of x

1. Prove the operator eqns

$$Dg = g_x + gD, \qquad D^2 g = g_{xx} + 2g_x D + g D^2.$$

$\forall$ functions $f(x)$

$$Dgf \equiv D(gf) = g_x f + g f_x = g_x f + g Df = (g_x + gD) f \Rightarrow Dg = g_x + gD$$

$$D^2 gf = DD(gf) = D(g_x f + g f_x) = g_{xx} f + g_x f_x + g_x f_x + g f_{xx}$$

$$= (g_{xx} + 2g_x D + g D^2) f$$

$$\Longrightarrow \quad D^2 g = g_{xx} + 2g_x D + g D^2$$

2. Prove more generally that

$$D^n g = \sum_{m=0}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m}$$

$$D^n(g(x)f(x)) = (\partial_x + \partial_y)^n g(x) f(y)\Big|_{y=x} = \sum_{m=0}^{n} \binom{n}{m} \partial_x^m g(x)\, \partial_y^{n-m} f(y)\Big|_{y=x}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \partial_x^m g(x) \partial_x^{n-m} f(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m} f$$

$$\Rightarrow D^n g = \sum_{m=0}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m}$$

$\boxed{\text{Ex } 59}$ Let $D \equiv \frac{\partial}{\partial x}$ and

$$L(u) = D^2 + u \quad , \quad M(u) = -\left(4D^3 + 6uD + 3u_x\right).$$

Check that

$$L(u)_t + [L(u), M(u)] = u_t + 6uu_x + u_{xxx}.$$

$L(u)_t = u_t$. For the commutator, use

$$[D^n, g] = \sum_{m=0}^{n-1} \binom{n}{m} \frac{\partial^{n-m} g}{\partial x^{n-m}} D^m$$

and
$$[A, BC] = [A,B]C + B[A,C]$$
$$[AB, C] = [A,C]B + A[B,C].$$

$$-[M(u), L(u)] = \left[4D^3 + 6uD + 3u_x, D^2 + u\right]$$

$$= 4[D^3, u] + 6[uD, D^2] + 6[uD, u] + 3[u_x, D^2]$$

$$= 4\left(u_{xxx} + 3u_{xx}D + 3u_x D^2\right) + 6[u, D^2]D + 6u[D,u] - 3[D^2, u_x]$$

$$= 4\left(u_{xxx} + 3\cancel{u_{xx}D} + \cancel{3u_x D^2}\right) - 6\left(\cancel{u_{xx}} + \cancel{2u_x D}\right)D + 6uu_x - 3\left(u_{xxx} + \cancel{2u_{xx}D}\right)$$

$$= 6uu_x + u_{xxx}$$

So $\quad L(u)_t + [L(u), M(u)] = u_t + 6uu_x + u_{xxx}.$

Ex 61

1. Show that the operator $D \equiv \frac{\partial}{\partial x}$ is anti-hermitian wrt the inner product

$$\langle \psi_1, \psi_2 \rangle := \int_{-\infty}^{+\infty} dx \, \bar{\psi}_1(x) \psi_2(x)$$

on the space $L^2(\mathbb{R})$ of square integrable functions, that is $\langle \psi_1, D\psi_2 \rangle = -\langle D\psi_1, \psi_2 \rangle$ for all $\psi_1, \psi_2 \in L^2(\mathbb{R})$.

$$\langle \psi_1, D\psi_2 \rangle = \int_{-\infty}^{+\infty} dx \, \bar{\psi}_1(x) \frac{\partial}{\partial x} \psi_2(x) \underset{\text{by parts}}{=} \underbrace{\left[\bar{\psi}_1(x) \psi_2(x) \right]_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{+\infty} dx \left(\frac{\partial}{\partial x} \bar{\psi}_1(x) \right) \psi_2(x)$$

for square integrable functions ($\psi_1, \psi_2 \xrightarrow[x \to \pm\infty]{} 0$)

$$= - \int_{-\infty}^{+\infty} dx \, \overline{\frac{\partial}{\partial x} \psi_1} \cdot \psi_2 = - \langle D\psi_1, \psi_2 \rangle \equiv \langle D^+ \psi_1, \psi_2 \rangle$$

$$\Rightarrow \quad D^+ = -D.$$

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2. Show that $L(u) = D^2 + u$ is hermitian, given that u is real.

See lectures.

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3. Given a Lax pair $(L(u), M(u))$, show that the hermitian part of $M(u)$ commutes with $L(u)$ and therefore drops out of the Lax eqn $L(u)_t = [M(u), L(u)]$. $\quad (*)$

Take hermitian conjugate of the Lax eqn and use $L(u) = L(u)^+$, $t \in \mathbb{R}$:

$$L(u)^+_t = [L(u)^+, M(u)^+] \quad \Rightarrow \quad L(u)_t = [-M(u)^+, L(u)] \qquad (**)$$

Take $\frac{1}{2}((*) + (**))$ and $\frac{1}{2}((*) - (**))$:

$$L(u)_t = [M(u) - M(u)^+, L(u)] \quad , \qquad 0 = [M(u) + M(u)^+, L]$$

$$M(u) = \underbrace{\frac{M(u)+M(u)^\dagger}{2}}_{\substack{\text{hermitian} \\ \text{part of } M \\ \equiv M_h(u)}} + \underbrace{\frac{M(u)-M(u)^\dagger}{2}}_{\substack{\text{anti-hermitian} \\ \text{part of } M \\ \equiv M_{ah}(u)}}$$

So $\quad L(u)_t = 2\left[M_{ah}(u), L(u)\right], \quad \left[M_h(u), L(u)\right] = 0.$

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4. Now assume that $M(u)$ is anti-hermitian : $M(u)^\dagger = -M(u)$. Show that $\langle \psi_1, \psi_2 \rangle$ is independent of time if $\psi_i(x;t)$ evolve according to the equation $\partial_t \psi_i = M(u)\psi_i$.

$$\partial_t \langle \psi_1, \psi_2 \rangle = \langle \partial_t \psi_1, \psi_2 \rangle + \langle \psi_1, \partial_t \psi_2 \rangle = \langle M(u)\psi_1, \psi_2 \rangle + \langle \psi_1, M(u)\psi_2 \rangle$$

$$= \langle M(u)\psi_1, \psi_2 \rangle + \langle M(u)^\dagger \psi_1, \psi_2 \rangle.$$

If $M(u)^\dagger = -M(u)$, then

$$\partial_t \langle \psi_1, \psi_2 \rangle = \langle M(u)\psi_1, \psi_2 \rangle - \langle M(u)\psi_1, \psi_2 \rangle = 0.$$