PROBLEMS CLASS 7 – 1/3/2024

Ex 69

1. Find a ~~function $f(u,u_x,u_{xx})$ and~~ a functional

$$F[u] = \int_{-\infty}^{+\infty} dx\, f(u, u_x, u_{xx})$$

such that the equation

$$u_t = \frac{\partial}{\partial x}\frac{\delta F[u]}{\delta u}$$

is the same as the 5^{th} order KdV equation

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0$$

Need

$$30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = -\frac{\partial}{\partial x}\frac{\delta F[u]}{\delta u} \qquad (*)$$

$$\Rightarrow \frac{\delta F[u]}{\delta u} = -\int dx \left[30u^2u_x + \underbrace{20u_xu_{xx} + 10uu_{xxx}}_{\parallel} + u_{xxxxx}\right] \quad \leftarrow \text{indefinite integral}$$

$$= -\int dx\left[30u^2u_x + \overbrace{10u_xu_{xx} + 10(uu_{xx})_x} + u_{xxxxx}\right]$$

$$= -\left(10u^3 + 5u_x^2 + 10uu_{xx} + u_{xxxx}\right) + c \quad \leftarrow \text{integration constant} \quad (**)$$

(could be a fn of t)

We need to rewrite this as

$$\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}}$$

for some $f(u,u_x,u_{xx})$. We can rewrite $(**)$ as

$$\frac{\delta F[u]}{\delta u} = -(u_{xx})_{xx} - 10(uu_x)_x + (5u_x^2 - 10u^3 + c)$$

$$\Rightarrow \frac{\partial f}{\partial u_{xx}} = -u_{xx}, \quad \frac{\partial f}{\partial u_x} = 10uu_x, \quad \frac{\partial f}{\partial u} = 5u_x^2 - 10u^3 + c$$

$$\Rightarrow \quad f(u,u_x,u_{xx}) = -\frac{1}{2}u_{xx}^2 + 5uu_x^2 - \frac{5}{2}u^4 + cu + \cancel{d}$$

d=0

No extra integration constant because it would make $F[u] = \int_{-\infty}^{+\infty} dx\, f(u, u_x, u_{xx})$ diverge, (assuming as usual $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$).

In the following I'll set $c=0$, because we just need a functional $F[u]$ which does the job.

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b. Show that $F[u] = \int_{-\infty}^{+\infty} dx\, f$ is a conserved quantity if u evolves according to the standard 3rd order KdV equation. [HARD!]

(SKIP IN PROBLEMS CLASS. EXERCISE.)

$$f = -\frac{1}{2}u_{xx}^2 + 5uu_x^2 - \frac{5}{2}u^4 \equiv \rho \quad \text{charge density}$$

Need to show that $\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$ for some j s.t. $[j]_{-\infty}^{+\infty} = 0$.

$$\frac{\partial \rho}{\partial t} = \frac{\partial f}{\partial t} = -u_{xx}u_{xxt} + 5u_t u_x^2 + 10uu_x u_{xt} - 10u^3 u_t$$

$$\underset{(KdV)}{\text{"="}} \; u_{xx}\underbrace{(6uu_x + u_{xxx})_{xx}}_{-(u_t)_{xx}} - 5(u_x^2 - 2u^3)\underbrace{(6uu_x + u_{xxx})}_{-u_t} - 10uu_x\underbrace{(6uu_x + u_{xxx})_x}_{-(u_t)_x}$$

("=" means equal up to a total x-derivative of a function which vanishes as $x \to \pm\infty$)

$$\underset{\text{integrate by parts}}{\text{"="}} \; \overbrace{(6uu_x + u_{xxx})}^{=-u_t} \cdot \left[u_{xxxx} - 5(u_x^2 - 2u^3) + 10u_x^2 + 10uu_{xx}\right]$$

$$\underset{(KdV)}{=} (6uu_x + u_{xxx}) \cdot (u_{xxxx} + 5u_x^2 + 10uu_{xx} + 10u^3)$$

$$= 6uu_x u_{xxxx} + 30uu_x^3 + 60u^2u_xu_{xx} + \underbrace{60u^4u_x}_{\text{tot. der}} + \underbrace{u_{xxx}u_{xxxx}}_{\text{tot. der.}}$$

$$+ 5u_x^2u_{xxx} + 10uu_{xx}u_{xxx} + 10u^3u_{xxx}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} \text{"="} \underline{6uu_xu_{xxxx}} + 30uu_x^3 + 60u^2u_xu_{xx} + 5u_x^2u_{xxx} + 10uu_{xx}u_{xxx} + 10u^3u_{xxx}$$

Now rewrite the underlined term up to a total derivative :

$$"=" -6u_x^2 u_{xxx} - 6uu_{xx}u_{xxx} + 30uu_x^3 + 60u^2u_xu_{xx} + 5u_x^2u_{xxx}$$
$$+ 10uu_{xx}u_{xxx} + 10u^3u_{xxx}$$

$$= -u_x^2u_{xxx} + 2u(u_{xx}^2)_x + 30uu_x^3 + 30u^2(u_x^2)_x + 10u^3(u_{xx})_x$$

$$"=" \cancel{2u_xu_{xx}^2} - \cancel{2u_xu_{xx}^2} + 30uu_x^3 - 60uu_x^3 - 30u^2 \underbrace{u_xu_{xx}}_{=\frac{1}{2}(u_x^2)_x}$$

$$"=" -30uu_x^3 + 30uu_x^3 = 0.$$

Therefore $\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$ for some j s.t. $j \to 0$ as $x \to \pm\infty$.

$$\Rightarrow \frac{d}{dt}\int_{-\infty}^{+\infty} dx\, \rho = \int_{-\infty}^{+\infty} dx \frac{\partial \rho}{\partial t} = -\int_{-\infty}^{+\infty} dx \frac{\partial j}{\partial x} = -\left[j\right]_{-\infty}^{+\infty} = 0.$$

C. Show that $\int_{-\infty}^{+\infty} dx\, u$ is a conserved quantity if u evolves according to the 5^{th} order KdV equation. [EASY!]

$$u_t = \frac{\partial}{\partial x}\frac{\delta F[u]}{\delta u} \quad \text{from part 1.}$$

$$\Leftrightarrow \underbrace{\frac{\partial}{\partial t}(u)}_{\rho} + \frac{\partial}{\partial x}\underbrace{\left(-\frac{\delta F[u]}{\delta u}\right)}_{j} = 0 \quad \text{s.t. } [j]_{-\infty}^{+\infty} = 0.$$

$$\Rightarrow \frac{d}{dt}\int_{-\infty}^{+\infty} dx\, u = 0.$$

Ex 70 Consider the scattering data for the potential $V(x) = a\delta(x)$:

$$S = \begin{cases} \left\{ R(k) = \dfrac{a}{2ik-a} \right\} & \text{if } a \geq 0 \\ \left\{ R(k) = \dfrac{a}{2ik-a} \right\}, \left\{ \mu_1 = -\dfrac{a}{2},\ c_1 = \sqrt{-\dfrac{a}{2}} \right\} & \text{if } a < 0 \end{cases}$$

For each sign of a:

(a) Calculate

$$F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k) e^{-ik\xi} + \sum_{n=1}^{N} c_n^2 e^{\mu_n \xi}.$$

[Hint: close the integration contour by adding an infinite arc in the upper/lower half of the complex plane for k, and use Cauchy's residue theorem. Look it up if you need a reminder.]

(b) Solve the Marchenko eqn

$$K(x,z) + F(x+z) + \int_{-\infty}^{x} dy\, K(x,y) F(y+z) = 0$$

to determine the unknown $K(x,z)$ $\forall z \leq x$ (set $K(x,z) = 0$ for $x < z$).

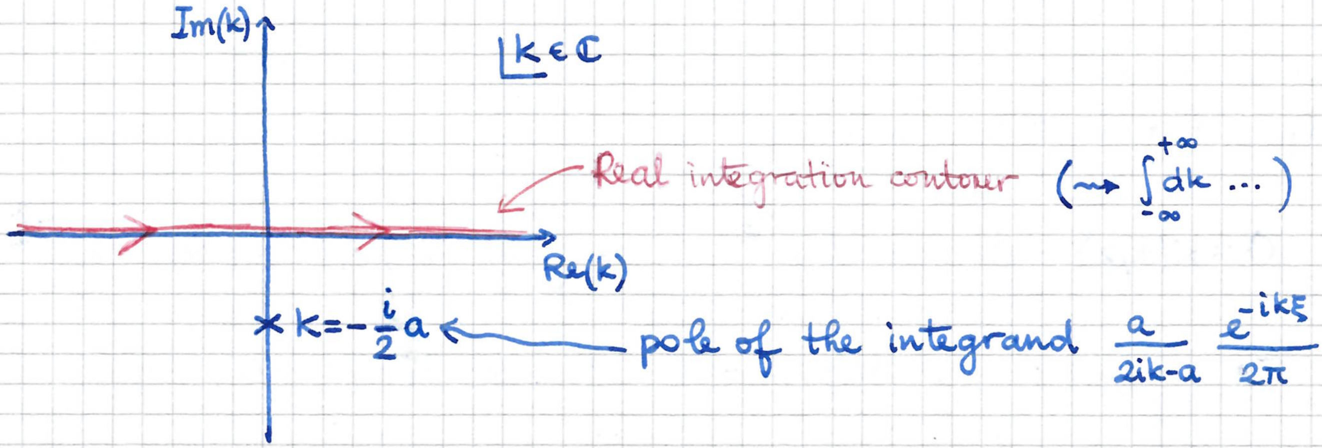
(c) Show that

$$V(x) = 2 \frac{d}{dx} K(x,x)$$

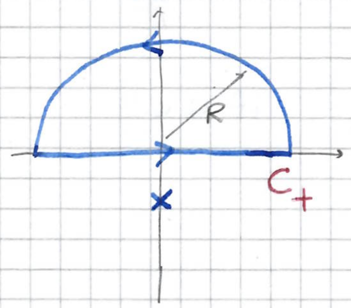
By this we really mean $\lim_{z \to x^-} K(x,z)$

WE'LL ONLY DO $a > 0$. ($a < 0$ in final assignment)
(no bound states)

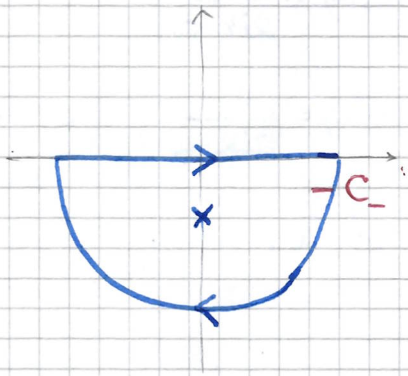
(a) $$F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{a}{2ik-a} e^{-ik\xi}$$



Following the hint, we'll close the integration contour by adding an arc at ∞ : $(R \to \infty)$



or



We choose the arc that doesn't contribute to the integral of

$$\frac{1}{2\pi}\frac{a}{2ik-a}\,e^{-ik\xi} \qquad \text{as} \quad R=|k| \longrightarrow \infty .$$

faster than the denominator diverges

- $\underline{\xi>0}$: $\left|e^{-ik\xi}\right| = e^{\xi\cdot \mathrm{Im}(k)} \longrightarrow 0$ along the lower infinite arc $(\mathrm{Im}\,k \to -\infty)$.

$$\Rightarrow F(\xi) = \oint_{-C_-} \frac{dk}{2\pi}\frac{a}{2ik-a}\,e^{-ik\xi} = -\frac{a}{2}\oint_{C_-}\frac{dk}{2\pi i}\frac{e^{-ik\xi}}{k+\frac{i}{2}a}$$

$$\underset{\text{(Cauchy)}}{=} -\frac{a}{2}\,\underset{k=-\frac{i}{2}a}{\mathrm{Res}}\frac{e^{-ik\xi}}{k+\frac{i}{2}a} = -\frac{a}{2}\lim_{k\to\frac{i}{2}a} e^{-ik\xi} = -\frac{a}{2}\,e^{-\frac{a}{2}\xi}$$

- $\underline{\xi<0}$: $\left|e^{-ik\xi}\right| \longrightarrow 0$ along the upper infinite arc.

$$\Rightarrow F(\xi) = \oint_{C_+}\frac{dk}{2\pi}\frac{a}{2ik-a}\,e^{-ik\xi} = 0$$ because C_+ doesn't enclose any poles.

$$\Rightarrow \quad \boxed{F(\xi) = -\frac{a}{2}\,e^{-\frac{a}{2}\xi}\,\Theta(\xi)}$$

Heaviside theta function $\quad \Theta(\xi) = \begin{cases} 1, & \xi>0 \\ (\text{say } \tfrac{1}{2} \text{ at } \xi=0) & \\ 0, & \xi<0 \end{cases}$

(b) Put in Marchenko eqn:

$$0 = K(x,z) - \frac{a}{2}\,\Theta(x+z)\,e^{-\frac{a}{2}(x+z)} + \int_{-\infty}^{x} dy\, K(x,y)\left(-\frac{a}{2}\right)\Theta(y+z)\,e^{-\frac{a}{2}(y+z)}$$

- $\underline{x+z<0}$: $\quad 0 = K(x,z) + 0 + 0 \;\Leftrightarrow\; K(x,z)=0.$

$\Downarrow$

$y+z<0$ in $\int$

• $\underline{x+z>0}$: $\quad 0 = K(x,z) - \frac{a}{2} e^{-\frac{a}{2}(x+z)} - \frac{a}{2}\int_{-z}^{x} dy\, K(x,y)\, e^{-\frac{a}{2}(y+z)}$

(and $x-z>0$, recall)

Try $K(x,z) = \alpha = \text{const.}$ to be determined.

$$0 = \cancel{\alpha} - \frac{a}{2} e^{-\frac{a}{2}(x+z)} + \alpha\left(e^{-\frac{a}{2}(x+z)} \cancel{-1}\right)$$

$$\Rightarrow \quad \alpha = \frac{a}{2}\,.$$

Hence $\quad K(x,z) = \frac{a}{2}\,\Theta(x+z)\,.$

(To be precise, we also set $K(x,z)=0$ for $x<z$, hence

$$\boxed{K(x,z) = \frac{a}{2}\,\underbrace{\Theta(x+z)\,\Theta(x-z)}_{=\mathbb{1}_{[-|z|,+|z|]}}}\,.$$)

"indicator function"

(c)

$$2\frac{d}{dx}\lim_{z\to x^-} K(x,z) = 2\frac{d}{dx}\left(\frac{a}{2}\Theta(2x)\right)$$

$$= 2\cdot\frac{a}{2}\frac{d}{dx}\Theta(x) = a\,\delta(x) = V(x) \quad \checkmark$$

$\uparrow$ $\Theta(2x) = \Theta(x)$ $\qquad$ $\uparrow$ $\frac{d}{dx}\Theta(x) = \delta(x)$