PROBLEMS CLASS 8 15/3/2024

$\boxed{\text{Ex 74}}$ If

1)
$$L(t) = \begin{pmatrix} x(t) & y(t) \\ y(t) & -x(t) \end{pmatrix},$$

find an antisymmetric matrix $M(t)$ s.t. the Lax equation $\dot{L}+[L,M]=0$ is equivalent to the system of ODE's

$$\begin{cases} \dot{x} = gy \\ \dot{y} = -gx \end{cases}$$

where $g(x,y,t)$ is some function of x,y,t.

2) Using only the symmetry properties of L, together with the Lax eqn, show that the eigenvalues of L do not depend on t.

3) Deduce the (otherwise fairly obvious) fact that if $x(t), y(t)$ evolve according to the above system of ODE's, then the value of $x(t)^2+y(t)^2$ remains constant.

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1) Let
$$M(t) = \begin{pmatrix} 0 & m(x,y,t) \\ -m(x,y,t) & 0 \end{pmatrix} = m\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be a general 2×2 antisymmetric real matrix.

$$0 = \dot{L}+[L,M] = \begin{pmatrix} \dot{x} & \dot{y} \\ \dot{y} & -\dot{x} \end{pmatrix} + \begin{pmatrix} x & y \\ y & -x \end{pmatrix}\begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} + \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$$

$$= \begin{pmatrix} \dot{x} & \dot{y} \\ \dot{y} & -\dot{x} \end{pmatrix} + m\left(\begin{pmatrix} -y & x \\ x & y \end{pmatrix} - \begin{pmatrix} y & -x \\ -x & -y \end{pmatrix}\right)$$

$$= \begin{pmatrix} \dot{x} & \dot{y} \\ \dot{y} & -\dot{x} \end{pmatrix} - 2m\begin{pmatrix} y & -x \\ -x & -y \end{pmatrix}$$

$$\iff \begin{cases} \dot{x} = 2my \\ \dot{y} = -2mx \end{cases} \qquad \text{hence} \quad g = 2m.$$

$$M(t) = \frac{g(x,y,t)}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

[There are many ways to answer, here's one (reminder of chapter 10)]

2) L is real symmetric, therefore it has real eigenvalues: $\lambda \in \mathbb{R}$

$(L^T = L = L^\dagger)$

$$L\psi = \lambda\psi \qquad \text{(all functions of } t\text{)}$$

$$\frac{d}{dt}: \quad \dot{L}\psi + L\dot{\psi} = \dot{\lambda}\psi + \lambda\dot{\psi}$$

$$\Rightarrow \dot{\lambda}\psi = (L-\lambda)\dot{\psi} + \dot{L}\psi$$

$$\underset{\text{(Lax eqn)}}{=} (L-\lambda)\dot{\psi} + ML\psi - LM\psi$$

$$\underset{L\psi=\lambda\psi}{=} (L-\lambda)(\dot{\psi} + M\psi)$$

Take inner product w/ ψ (i.e. $\underset{\text{row}}{\psi^\dagger}(\underset{\text{column}}{\top})$):

$$\dot{\lambda}\,\psi^\dagger\psi = \psi^\dagger(L-\lambda)(\dot{\psi} + M\psi)$$

$$= \left((L-\lambda)^\dagger\psi\right)^\dagger(\dot{\psi} + M\psi)$$

$$\underset{L=L^\dagger,\,\lambda\in\mathbb{R}}{=} \left((L-\lambda)\psi\right)^\dagger(\dot{\psi} + M\psi)$$

$$\underset{(L-\lambda)\psi=0}{=} 0.$$

$\psi \neq 0 \Rightarrow \psi^\dagger\psi = \|\psi\|^2 \neq 0$. Then $\dot{\lambda} = 0$.

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3) $$\frac{d}{dt}\operatorname{tr}(L^2) = \operatorname{tr}(\dot{L}L + L\dot{L}) = 2\operatorname{tr}([M,L]L)$$

$$= 2\operatorname{tr}(ML^2) - 2\operatorname{tr}(LML) = 0$$

by cyclicity of the trace.

Now calculate

$$\operatorname{tr}(L^2) = \operatorname{tr}\left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix}\begin{pmatrix} x & y \\ y & -x \end{pmatrix}\right) = \operatorname{tr}\left(\begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix}\right)$$

$$= 2(x^2+y^2)$$

Hence $x(t)^2 + y(t)^2$ does not depend on t.

$\boxed{\text{Ex } 75}$

Let $q_{i+3} \equiv q_i$ and $p_{i+3} \equiv p_i$. A Lax pair of matrices L and M are given by

$$L = \begin{pmatrix} p_1 & b_1 & b_3 \\ b_1 & p_2 & b_2 \\ b_3 & b_2 & p_3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & b_1 & -b_3 \\ -b_1 & 0 & b_2 \\ b_3 & -b_2 & 0 \end{pmatrix}$$

where $\dot{p}_i = q_i$ and $b_i = \exp\left[c(q_i - q_{i+1})\right]$ for some real constant c. Use the Lax equation $\dot{L} + [L, M] = 0$ to find the constant c and to obtain equations of motion in the form $\ddot{q}_i = f_i(q)$, for some functions $f_i(q)$ that you should determine.

$\left[\dot{g} = \frac{dg}{dt}, \quad \ddot{g} = \frac{d^2 g}{dt^2} \quad \text{etc.}\right]$

$$\dot{L} = \begin{pmatrix} \dot{p}_1 & \dot{b}_1 & \dot{b}_3 \\ \dot{b}_1 & \dot{p}_2 & \dot{b}_2 \\ \dot{b}_3 & \dot{b}_2 & \dot{p}_3 \end{pmatrix}$$

$$[M, L] = \begin{pmatrix} 2(b_1^2 - b_3^2) & b_1 p_2 - \cancel{b_2 b_3} - b_1 p_1 + \cancel{b_2 b_3} & \cancel{b_1 b_2} - b_3 p_3 + b_3 p_1 - \cancel{b_1 b_2} \\ b_1(p_2 - p_1) & 2(b_2^2 - b_1^2) & -\cancel{b_1 b_3} + b_2 p_3 + \cancel{b_1 b_3} - b_2 p_2 \\ b_3(p_1 - p_3) & b_2(p_3 - p_2) & 2(b_3^2 - b_2^2) \end{pmatrix}$$

where the red entries have been worked out using the fact that

$L = L^T, \quad M = -M^T \quad \Rightarrow \quad [M, L] = [M, L]^T$.

$\boxed{\text{Entries } i+1, i}$ (i defined mod 3)

$$\dot{b}_i = b_i(p_{i+1} - p_i) \quad \Leftrightarrow \quad \dot{(\ln b_i)} = p_{i+1} - p_i$$

Plug in $b_i = e^{c(q_i - q_{i+1})}$, $\dot{q}_i = p_i$: $\quad c(p_i - p_{i+1}) = p_{i+1} - p_i$

$$\Rightarrow \quad c = -1$$

$\boxed{\text{Entries } i,i}$

$$\dot{p}_i = 2\left(b_i^2 - b_{i-1}^2\right)$$

Plug in $p_i = \dot{q}_i$, $b_i = e^{q_{i+1} - q_i}$:

$$\boxed{\ddot{q}_i = 2\left(e^{2(q_{i+1}-q_i)} - e^{2(q_i - q_{i-1})}\right) \equiv f_i(q).}$$

BONUS:

$$L = \begin{pmatrix} p_1 & b_1 & & & & & b_n \\ b_1 & p_2 & b_2 & & & & \\ & b_2 & p_3 & b_3 & & \text{\huge 0} & \\ & & b_3 & p_4 & & & \\ & \text{\huge 0} & & & \ddots & & \\ & & & & & p_{n-1} & b_{n-1} \\ b_n & & & & & b_{n-1} & p_n \end{pmatrix}, \quad M = \begin{pmatrix} 0 & b_1 & & & & & -b_n \\ -b_1 & 0 & b_2 & & & \text{\huge 0} & \\ & -b_2 & 0 & b_3 & & & \\ & & -b_3 & 0 & & & \\ & \text{\huge 0} & & & \ddots & & \\ & & & & & 0 & b_{n-1} \\ b_n & & & & & -b_{n-1} & 0 \end{pmatrix}$$

Calculate $[L, M]$.

Use $(i \equiv i+n)$

$$L_{ij} = p_i \delta_{i,j} + b_j \delta_{i,j+1} + b_i \delta_{i,j-1}$$

$$M_{ij} = -b_j \delta_{i,j+1} + b_i \delta_{i,j-1}$$

$$(LM)_{ij} = \sum_k L_{ik} M_{kj}$$

$$= \sum_k \left(p_i \delta_{i,k} + b_k \delta_{i,k+1} + b_i \delta_{i,k-1}\right)\left(-b_j \delta_{k,j+1} + b_k \delta_{k,j-1}\right)$$

$$= \sum_k \Big(-p_i b_{\overset{i-1}{\overset{\uparrow}{j}}} \delta_{i,k} \delta_{k,j+1} + p_i b_{\overset{i}{\overset{\uparrow}{k}}} \delta_{i,k} \delta_{k,j-1} - b_{\overset{i-1}{\overset{\uparrow}{k}}} b_{\overset{i-2}{\overset{\uparrow}{j}}} \delta_{i,k+1} \delta_{k,j+1}$$

$$+ b^2_{\underset{j-1}{\underset{\downarrow}{k}}} \delta_{i,k+1} \delta_{k,j-1} - b_i b_{\underset{i}{\underset{\downarrow}{j}}} \delta_{i,k-1} \delta_{k,j+1} + b_i b_{\underset{i+1}{\underset{\downarrow}{k}}} \delta_{i,k-1} \delta_{k,j-1}\Big)$$

$$= p_i b_i \delta_{i,j-1} - p_i b_{i-1} \delta_{i,j+1} + (b_{i-1}^2 - b_i^2)\delta_{i,j} + (b_i b_{i+1} \delta_{i,j-2} - b_j b_{j+1} \delta_{i,j+2})$$

Similarly,

$$(ML)_{ij} = b_i p_{i+1} \delta_{i,j-1} - p_{i-1} b_{i-1} \delta_{i,j+1} + (-b_{i-1}^2 + b_i^2)\delta_{i,j} + (b_i b_{i+1} \delta_{i,j-2} - b_j b_{j+1} \delta_{i,j+2})$$

$$\Rightarrow [L, M]_{ij} = 2(b_{i-1}^2 - b_i^2)\delta_{i,j} + b_i(p_i - p_{i+1})\delta_{i,j-1} - b_{i-1}(p_i - p_{i-1})\delta_{i,j+1}$$