PROBLEMS CLASS 1 — 26/10/2022

$\boxed{\text{Ex } 13.2}$

Find (if they exist) real non-singular travelling wave sol'ns of

"wrong sign" mKdV : $\quad u_t - 6u^2 u_x + u_{xxx} = 0$

BC : $\quad u, u_x, u_{xx} \to 0 \quad$ as $|x| \to \infty \quad \forall t$

Sub in trav. wave $\quad u(x,t) = f(x - \overset{\text{velocity}}{v} t) \equiv f(\xi)$:

$$-vf' - 6f^2 f' + f''' = 0$$

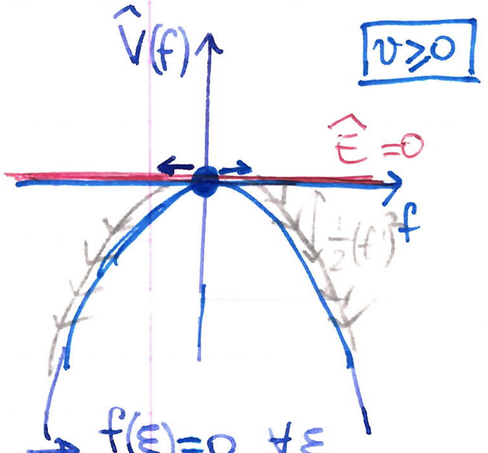
$$\xrightarrow[\int d\xi]{} \quad -vf - 2f^3 + f'' = A \quad \leftarrow \text{const} \qquad (1)$$

$$\xrightarrow[\int d\xi\, f']{} \quad -\frac{v}{2}f^2 - \frac{1}{2}f^4 + \frac{1}{2}(f')^2 = Af + B \qquad (2)$$

BC $\left(f, f', f'' \xrightarrow[|\xi|\to\infty]{} 0\right)$: $\quad A = B = 0$

Rearrange (2): $\quad \underbrace{\frac{1}{2}(f')^2}_{\text{"kin. en." } \geq 0} + \underbrace{\left(-\frac{f^4}{2} - \frac{v}{2}f^2\right)}_{\hat{V}(f) \text{ "Pot. en."}} = \underbrace{0}_{\hat{E}=0 \text{ "Tot. energy"}}$

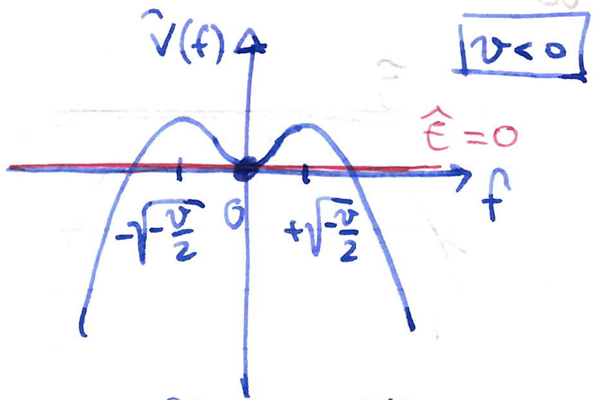
1d point particle analogy



$f(\xi) = 0 \quad \forall \xi$

trivial solution

(all other sol'ns are singular)



$f(\xi) = 0 \quad \forall \xi$

trivial solution

$\boxed{\text{Ex } 13.4}$ Same for

"ϕ^6 theory" : $u_{tt} - u_{xx} + u(u^2-1)(3u^2-1) = 0$

BC : $\begin{cases} u_t, u_x, u \longrightarrow 0 & x \to -\infty \\ u_t, u_x, u-1 \longrightarrow 0 & x \to +\infty \end{cases}$

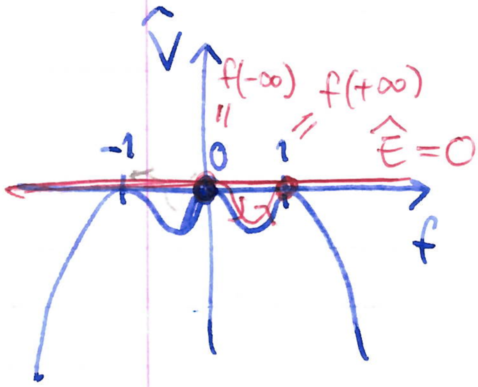
Sub in $u(x,t) = f(x-vt) \equiv f(\xi)$

$$(v^2-1)\, f'' + f(f^2-1)(3f^2-1) = 0$$

$$\Leftrightarrow \quad f'' - \gamma^2 \underbrace{f(f^2-1)(3f^2-1)}_{\equiv 3f^5 - 4f^3 + f} = 0 \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\underset{\int d\xi\, f'}{\Longrightarrow} \quad \underbrace{\tfrac{1}{2}(f')^2}_{\substack{|\xi|\to\infty \\ \downarrow \\ 0}} - \gamma^2 \underbrace{\left(\frac{f^6}{2} - f^4 + \frac{f^2}{2}\right)}_{\substack{\downarrow \\ 0}} = \overset{\text{const}}{A} \underset{\substack{\uparrow \\ \text{BC}}}{=} 0$$

$$\tfrac{1}{2}(f')^2 \underbrace{- \tfrac{1}{2}\gamma^2 f^2 (f^2-1)^2}_{\hat{V}(f)} = \underset{\hat{E}}{\overset{\shortparallel}{0}}$$



BC : $f' \to 0$ as $|\xi| \to \infty$

$\underset{\xi\to-\infty}{f \to 0}$, $\underset{\xi\to+\infty}{f \to 1}$

A real non-singular sol'n exists. Let's find it!

$$f' = \pm \gamma f (f^2 - 1)$$

Need $0 < f < 1$ and $f' > 0$, so

$$f' = \gamma f (1 - f^2)$$

$$\int \frac{df}{f(1-f^2)} = \gamma \int d\xi$$

$$\gamma \int d\xi = \gamma(\xi - x_0) = \gamma(x - x_0 - vt)$$

partial fractions $\|$

$$\int df \left(\frac{1}{f} + \frac{1}{2}\frac{1}{1-f} - \frac{1}{2}\frac{1}{1+f} \right) = \ln f - \frac{1}{2}\ln(1-f) - \frac{1}{2}\ln(1+f)$$

$$= \ln \frac{f}{\sqrt{1-f^2}}$$

$$\Rightarrow \ln \frac{f}{\sqrt{1-f^2}} = \gamma(x - x_0 - vt)$$

$$\frac{f}{\sqrt{1-f^2}} = e^{\gamma(x-x_0-vt)}$$

$$\frac{f^2}{1-f^2} = e^{2\gamma(x-x_0-vt)}$$

$$\Rightarrow f^2 = \frac{e^{2\gamma(x-x_0-vt)}}{1 + e^{2\gamma(x-x_0-vt)}} = \frac{1}{e^{-2\gamma(x-x_0-vt)} + 1}$$

$$u(x,t) = f(x - vt) = \cancel{\pm} \frac{1}{\sqrt{e^{-2\gamma(x-x_0-vt)} + 1}}$$

PROBLEMS CLASS 2 - 9/11/2022

$\boxed{\text{Ex 17}}$

In this exercise you will learn how to generalise the Bogomol'nyi bound to field configurations with |topological charge| > 1.

① Explain why the Bogom. argument given in the lectures fails to provide a useful bound on the energy of a solution of the S-G eqn w/ topological charge $n_+ - n_- = 2$. What's the most that can be said about the energy of a field with $n_+ - n_- = k$?

- Bogomol'nyi:

$$E = \int_{-\infty}^{+\infty} dx \left[\underset{\geq 0}{\frac{1}{2}u_t^2} + \underset{\geq 0}{\frac{1}{2}u_x^2} + \underset{\geq 0}{\overbrace{1-\cos u}^{=2\sin^2\frac{u}{2}}} \right]$$

$$= \int_{-\infty}^{+\infty} dx \left[\underset{\geq 0}{\frac{1}{2}u_t^2} + \underset{\geq 0}{\frac{1}{2}\left(u_x \pm 2\sin\frac{u}{2}\right)^2} \right] \pm 4\left[\cos\frac{u}{2}\right]_{-\infty}^{+\infty}$$

$$\geq 4\left|\left[\cos\frac{u}{2}\right]_{-\infty}^{+\infty}\right|$$

- $u(\pm\infty, t) = 2\pi n_\pm$ with $k = n_+ - n_-$.

$$\left[\cos\frac{u}{2}\right]_{-\infty}^{+\infty} = \cos(\pi n_+) - \cos(\pi n_-) = (-1)^{n_+} - (-1)^{n_-}$$

$$= (-1)^{n_-}\left[(-1)^{n_+ - n_-} - 1\right] = (-1)^{n_-}\left[(-1)^k - 1\right] \leftarrow$$

For $\underline{k=2}$,

$$E \geq 4\left|\left[\cos\frac{u}{2}\right]_{-\infty}^{+\infty}\right| = 0 \quad \text{which we already knew.}$$

For $\underline{\text{general } k}$,

$$E \geq 4\left|(-1)^k - 1\right| = \begin{cases} 0, & k \text{ is even} \quad (\text{same as } k=0,\pm 2) \\ 8, & k \text{ is odd} \quad (\text{same as } k=\pm 1) \end{cases}$$

② For a sine-Gordon field u, generalise the Bogomol'nyi argument to show that

$$\int_A^B dx\left[\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+(1-\cos u)\right] \geq \pm 4\left[\cos\frac{u}{2}\right]_A^B .$$

$$\int_A^B dx\left[\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+2\sin^2\frac{u}{2}\right]$$

$$= \int_A^B dx\left[\frac{1}{2}u_t^2+\frac{1}{2}\left(u_x\pm 2\sin\frac{u}{2}\right)^2 \overbrace{\mp 2u_x\sin\frac{u}{2}}^{\text{total } x\text{-derivative}}\right]$$

$$= \int_A^B dx\left[\underbrace{\frac{1}{2}u_t^2}_{\geq 0}+\underbrace{\frac{1}{2}\left(u_x\pm 2\sin\frac{u}{2}\right)^2}_{\geq 0}\right] \pm 4\left[\cos\frac{u}{2}\right]_A^B \geq \pm 4\left[\cos\frac{u}{2}\right]_A^B .$$

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③ Use this result and the intermediate value theorem to show that if the field u has the boundary conditions of a k-kink (that is, $n_+-n_-=k$), then its energy is at least $|k|$ times that of a single kink. Can this bound be saturated?

We would like to show that $E \geq 8|k|$. top. charge

Consider a sol'n w/ $k=n_+-n_-$, and let's set $u(-\infty,t)=2\pi n_-\equiv 0$ wlog by shifting $u(x,t)$ by an appropriate integer multiple of 2π:

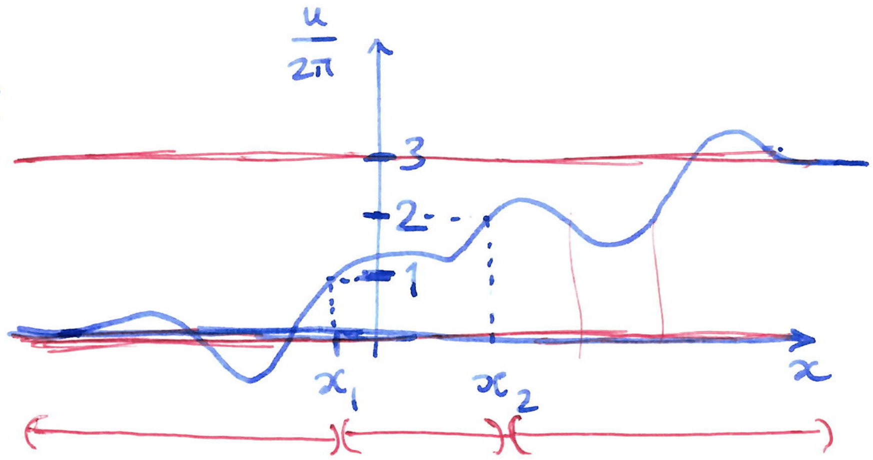
$$u(-\infty,t)=0\ , \quad u(+\infty,t)=2\pi k .$$

We know (/assume) that u is continuous. By the IVT, all values in $(0,2\pi k)$ are taken by the field u at least once as $x\in\mathbb{R}$. ($\forall t$, keep fixed)

(Assume $k>0$ from now on.)

Let's call $x_1 < x_2 < \dots < x_{k-1}$ the smallest values of x s.t. $u(x_n,t) = 2\pi n$.

E.g. for $k=3$ (fixed t)



$$E = \int_{-\infty}^{+\infty} dx\, \mathcal{E} = \int_{-\infty}^{x_1} dx\, \mathcal{E} + \int_{x_1}^{x_2} dx\, \mathcal{E} + \int_{x_2}^{x_3} dx + \dots + \int_{x_{k-1}}^{+\infty} dx\, \mathcal{E}$$

where $\mathcal{E} = \frac{1}{2}\left[u_t^2 + u_x^2 + 4\sin^2\frac{u}{2}\right]$.

Now apply the Bogo bound of part 2 interval by interval:

$$E \geq -4s_1\left[\cos\frac{u}{2}\right]_{-\infty}^{x_1} - 4s_2\left[\cos\frac{u}{2}\right]_{x_1}^{x_2} \dots - 4s_{k-1}\left[\cos\frac{u}{2}\right]_{x_{k-2}}^{x_{k-1}} - 4s_k\left[\cos\frac{u}{2}\right]_{x_{k-1}}^{+\infty}$$

where $s_n = \pm 1$ is the sign appearing in the Bogo eqn

$$\boxed{u_x = 2s_n \sin\frac{u}{2}} \quad \text{for } x \in (x_{n-1}, x_n). \quad (*) \quad \begin{pmatrix} x_0 \equiv -\infty \\ x_k \equiv +\infty \end{pmatrix}$$

$$= -4\underset{+1}{s_1}\overset{-2}{(-1-1)} - 4\underset{-1}{s_2}\overset{+2}{(1-(-1))} - 4\underset{+1}{s_3}\overset{-2}{(-1-1)} \dots \qquad (k \text{ terms})$$

Pick alternating signs $s_n = (-1)^{n-1}$, so that $\boxed{E \geq 8k}$.

- The bound can be saturated iff $u_t = 0$ and $(*)$ holds for all $x \in (x_{n-1}, x_n)$ for all n. The solution is a static kink or antikink, which only tends to integer multiples of 2π as $x \to \pm\infty$. That contradicts that $u(x_n,t) = 2\pi n$ for finite x_n. So the Bogomol'nyi eqns $(*)$ cannot all be satisfied simultaneously. $\Rightarrow E > 8k$ if $k>1$.

PROBLEMS CLASS 3 – 23/11/2022

Ex 23

Consider the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$.

1. Show that $\rho_1 \equiv u$, $\rho_2 \equiv u^2$ and $\rho_* \equiv xu - 3tu^2$ are all conserved densities, so that

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_* = \int_{-\infty}^{+\infty} dx\ (xu - 3tu^2) \tag{4.7}$$

are all conserved charges. seen in lectures Do as an exercise!

2. Evaluate the conserved charges Q_1, Q_2 and Q_* for the one-soliton solution centred at x_0 and moving with velocity $v = 4\mu^2$:

$$u_{\mu,x_0}(x,t) = 2\mu^2 \text{sech}^2\left[\mu(x - x_0 - \underbrace{4\mu^2}_{v} t)\right] \ . \tag{4.8}$$

$$\left[\int_{-\infty}^{+\infty} dx\, \text{sech}^2 x = 2\ , \quad \int_{-\infty}^{+\infty} dx\, \text{sech}^4 x = \frac{4}{3}\right]$$

$y = \mu(x - x_0 - vt)$
$dy = \mu\, dx$

$$Q_1 = 2\mu^2 \int_{-\infty}^{+\infty} dx\ \text{sech}^2[\mu(x - x_0 - vt)] = 2\mu \int_{-\infty}^{+\infty} dy\ \text{sech}^2 y = 4\mu\ .$$

$$Q_2 = 4\mu^4 \int_{-\infty}^{+\infty} dx\ \text{sech}^4[\mu(x - x_0 - vt)] = 4\mu^3 \int_{-\infty}^{+\infty} dy\ \text{sech}^4 y = \frac{16}{3}\mu^3\ .$$

$$Q_* = 2\mu^2 \int_{-\infty}^{+\infty} dx\ x\, \text{sech}^2[\mu(x - x_0 - vt)] - 3t\, Q_2$$

$x \to (x - x_0 - vt) + (x_0 + vt)$

$$= 2\int_{-\infty}^{+\infty} dy\ \underbrace{y\, \text{sech}^2 y}_{\text{odd}} + (x_0 + vt)\, Q_1 - 3t\, Q_2$$

$= 0$

$$= (x_0 + \cancel{4\mu^2 t}) \cdot 4\mu - \cancel{3t\, \frac{16}{3}\mu^3} = 4\mu x_0\ .$$

3. According to the KdV equation, the initial condition $\overset{t=0}{u(x,0) = 6\,\mathrm{sech}^2(x)}$ is known to evolve into the sum of two well-separated solitons with different velocities $v_1 = 4\mu_1^2$ and $v_2 = 4\mu_2^2$ at late $\overset{t\to\infty}{\text{times}}$. Use the conservation of Q_1 and Q_2 to determine v_1 and v_2.

$$Q_{\substack{1\\2}}\,(t=0) = Q_{\substack{1\\2}}\,(t\to\infty)$$

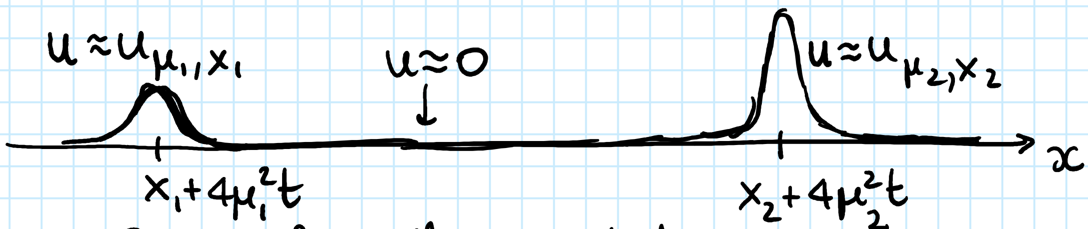
• $\underline{t=0}$: $\quad u(x,0) = 6\,\mathrm{sech}^2 x$

$$Q_1 = \int_{-\infty}^{+\infty} dx\; 6\,\mathrm{sech}^2 x = 6\cdot 2 = 12$$

$$Q_2 = \int_{-\infty}^{+\infty} dx\; 36\,\mathrm{sech}^4 x = 36\cdot\frac{4}{3} = 48\,.$$

• $\underline{t\to+\infty}$: $\quad u(x,t) \approx u_{\mu_1,x_1}(x,t) + u_{\mu_2,x_2}(x,t)$

where $\quad u_{\mu,x_0}(x,t) = 2\mu^2\,\mathrm{sech}^2\left[\mu(x-x_0-4\mu^2 t)\right].$



Fixed $t\to\infty$

Sum of well separated solitons.

$$Q_1 = \int_{-\infty}^{+\infty} dx\, u(x,t) \approx \int_{-\infty}^{+\infty} dx\, u_{\mu_1,x_1}(x,t) + \int_{-\infty}^{+\infty} dx\, u_{\mu_2,x_2}(x,t) = 4\mu_1 + 4\mu_2$$

$$Q_2 = \int_{-\infty}^{+\infty} dx\, u(x,t)^2 = \int_{-\infty}^{+\infty} dx\, u^2_{\mu_1,x_1}(x,t) + \int_{-\infty}^{+\infty} dx\, u^2_{\mu_2,x_2}(x,t) + 2\int_{-\infty}^{+\infty} dx\, \underbrace{u_{\mu_1,x_1} u_{\mu_2,x_2}}_{\approx 0}$$

$$= \frac{16}{3}(\mu_1^3+\mu_2^3)\,.$$

Equate results at $t=0$ & $t\to+\infty$:

$$\begin{cases} \cancel{12}^{\,3} = \cancel{4}(\mu_1+\mu_2) \\ \cancel{48}^{\,3} = \frac{\cancel{16}}{3}(\mu_1^3+\mu_2^3) \end{cases} \Rightarrow \begin{cases} \mu_1+\mu_2 = 3 \\ \cancel{(\mu_1+\mu_2)}(\mu_1^2-\mu_1\mu_2+\mu_2^2) = \cancel{9}^{\,3} \end{cases}$$

$$\mu_1^2-\mu_1\mu_2+\mu_2^2 = (\mu_1+\mu_2)^2 - 3\mu_1\mu_2 = 9-3\mu_1\mu_2$$

$$\begin{cases} \mu_1+\mu_2 = 3 \\ \mu_1\mu_2 = 2 \end{cases}$$

$$\Rightarrow (\mu_1,\mu_2) = (1,2) \text{ or } (2,1) \Rightarrow$$

Velocities $v_i = 4\mu_i^2$

$= 4,\ 16\,.$

4. A two-soliton solution separates as $t \to -\infty$ into two one-solitons u_{μ_1, x_1} and u_{μ_2, x_2}. As $t \to +\infty$, two one-solitons are again found, with μ_1 and μ_2 unchanged but with x_1, x_2 replaced by y_1, y_2. Use the conservation of Q_* to find a formula relating the *phase shifts* $y_1 - x_1$ and $y_2 - x_2$ of the two solitons.

Please complete the exercise.

Ex 28

1. Show that the two equations

$$\begin{cases} v_x = -u - v^2 & \text{(a)} \\ v_t = 2u^2 + 2uv^2 + u_{xx} - 2u_x v & \text{(b)} \end{cases} \tag{5.4}$$

are a Bäcklund transform relating solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \text{(kdV)} \tag{5.5}$$

and the wrong sign modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0\,. \qquad \text{(wsmKdV)} \tag{5.6}$$

(Note the appearance of the Miura transform in (5.4).)

- To find the eqn for v, solve (a) for u: $u = -(v_x + v^2)$.
  Sub in (b):

$$v_t = 2(v_x+v^2)^2 - 2(v_x+v^2)v^2 - \overbrace{(v_x+v^2)_{xx}}^{v_{xxx}+2vv_{xx}+2v_x^2} + 2\overbrace{(v_x+v^2)_x}^{v_{xx}+2vv_x} v$$

$$= 2(\cancel{v_x^2} + \cancel{v^4} + \underline{2v_xv^2} - \underline{v_xv^2} - \cancel{v^4}) - v_{xxx} - \cancel{2vv_{xx}} - \cancel{2v_x^2} + \cancel{2v_{xx}v} + \underline{4v^2v_x}$$

$$= 6v_xv^2 - v_{xxx} \qquad (5.6) \quad ✓$$

- To find the eqn for u, cross-differentiate:

$(a)_t$: $$v_{xt} = -u_t - 2vv_t \overset{(b)}{=} -u_t - 2v(2u^2 + 2uv^2 + u_{xx} - 2u_xv) \qquad (*)$$

$(b)_x$: $$v_{tx} = 4uu_x + 2u_xv^2 + 4uv v_x + u_{xxx} - 2u_{xx}v - 2u_x v_x \qquad (**)$$

$$\overset{(a)}{=} 4uu_x + 2u_xv^2 + u_{xxx} - 2u_{xx}v + 2(2uv - u_x)(-u-v^2)$$

Compare:

$$-u_t - 2v(\cancel{2u^2 + 2uv^2 + u_{xx}} - 2u_xv)$$

$$= 6uu_x + u_{xxx} + 2v(u_xv \cancel{- u_{xx} - 2u^2 - 2uv^2} + u_xv)$$

$$\Rightarrow \quad u_t + 6uu_x + u_{xxx} = 0\,. \qquad (5.5) \quad ✓$$

Ex 28

1. Show that the two equations

$$\begin{aligned} v_x &= -u - v^2 \\ v_t &= 2u^2 + 2uv^2 + u_{xx} - 2u_x v \end{aligned} \tag{5.4}$$

are a Bäcklund transform relating solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5.5}$$

and the wrong sign modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0\,. \tag{5.6}$$

(Note the appearance of the Miura transform in (5.4).)

2. Taking $u = c^2$, where c is a constant, as a seed solution of the KdV equation, find the corresponding solution of the wrong sign mKdV equation.

Please attempt the exercise. The solution is below.

$$\begin{cases} v_x = -(c^2+v^2) & (\star) \\ v_t = 2c^4 + 2c^2v^2 = 2c^2(c^2+v^2) & (\star\star) \end{cases}$$

Solve $(\star)$:

$$\int dx = -\int \frac{dv}{c^2+v^2}$$

$$\Rightarrow x - f(t) = -\frac{1}{c}\int \frac{d(v/c)}{1+\left(\frac{v}{c}\right)^2} = -\frac{1}{c}\arctan\frac{v}{c}$$

Solve $(\star\star)$:

$$\int dt = \frac{1}{2c^2}\int \frac{dv}{c^2+v^2}$$

$$\Rightarrow t - g(x) = \frac{1}{2c^3}\arctan\frac{v}{c}$$

$$\arctan\frac{v}{c} = -c\left(x - f(t)\right) = 2c^3\left(t - g(x)\right)$$

The 2nd equality implies

$$-cx + 2c^3 g(x) = 2c^3 t - cf(t) = \text{const} \equiv -cx_0$$

$$\Rightarrow \quad f(t) = 2c^2 t + x_0$$

$$\Rightarrow \quad \arctan \frac{v}{c} = -c(x - x_0 - 2c^2 t)$$

$$\Rightarrow \quad v = -c \tan\left[c(x - x_0 - 2c^2 t)\right] \quad \leftarrow \text{solution of wsmKdV eqn}$$

NOTE: this is a singular solution if $c \in \mathbb{R}$.

If however c is purely imaginary, $c = id$ ($d \in \mathbb{R}$), then using $\tan(ix) = i \tanh x$ the solution becomes

$$v = d \cdot \tanh\left[d(x - x_0 + 2d^2 t)\right],$$

which is a regular solution with velocity $-2d^2 \leq 0$.

PROBLEMS CLASS 4 – 7/12/2022

Ex 36

The Hirota bilinear differential operator $D_t^m D_x^n$ is defined for any pair of natural numbers (m,n) by

$$D_t^m D_x^n(f,g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x\\t'=t}} \tag{6.3}$$

and maps a pair of functions $(f(x,t), g(x,t))$ into a single function.

1. Prove that the Hirota operators $B_{m,n} := D_t^m D_x^n$ are bilinear, *i.e.* for all constants a_1, a_2

$$\begin{aligned} B_{m,n}(a_1 f_1 + a_2 f_2, g) &= a_1 B_{m,n}(f_1,g) + a_2 B_{m,n}(f_2,g)\ , \\ B_{m,n}(f, a_1 g_1 + a_2 g_2) &= a_1 B_{m,n}(f,g_1) + a_2 B_{m,n}(f,g_2)\ . \end{aligned} \tag{6.4}$$

$$(\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n \underset{\text{bin thm}}{=} \sum_{h=0}^{m} \binom{m}{h} (-1)^{m-h} \partial_t^h \partial_{t'}^{m-h} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \partial_x^k \partial_{x'}^{n-k}$$

$$= \sum_{h=0}^{m} \sum_{k=0}^{n} \binom{m}{h}\binom{n}{k} (-1)^{m+n-h-k} \left(\partial_t^h \partial_x^k\right)\left(\partial_{t'}^{m-h} \partial_{x'}^{n-k}\right)$$

is a linear combination of linear diff. ops $\partial_t^h \partial_x^k$ in (t,x)
and of " " " $\partial_{t'}^{m-h}\partial_{x'}^{n-k}$ in (t',x'):

$$\partial_t^h \partial_x^k \left(a_1 f_1(t,x) + a_2 f_2(t,x)\right) = a_1 \partial_t^h \partial_x^k f_1(t,x) + a_2 \partial_t^h \partial_x^k f_2(t,x),$$

$$\partial_{t'}^{m-h} \partial_{x'}^{n-k} \left(a_1 g_1(t',x') + a_2 g_2(t',x')\right) = a_1 \partial_{t'}^{m-h} \partial_{x'}^{n-k} g_1(t',x') + a_2 \partial_{t'}^{m-h} \partial_{x'}^{n-k} g_2(t',x')$$

$$\Rightarrow \quad D_t^m D_x^n (a_1 f_1 + a_2 f_2, g) = a_1 D_t^m D_x^n (f_1, g) + a_2 D_t^m D_x^n (f_2, g)$$

and similarly for the 2nd eqn.

2. Prove the symmetry property

$$\overset{D_t^m D_x^n}{\cancel{B_{m,n}}}(f,g) = (-1)^{m+n} B_{m,n}(g,f)\ . \qquad (6.5)$$

$$\left[D_t^m D_x^n (f,g)\right](t,x) = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t,x)\, g(t',x')\Big|_{\substack{x'=x \\ t'=t}}$$

$$= (-1)^m (\partial_{t'} - \partial_t)^m (-1)^n (\partial_{x'} - \partial_x)^n g(t',x') f(t,x)\Big|_{\substack{x'=x \\ t'=t}}$$

$$= (-1)^{m+n} (\partial_{t'} - \partial_t)^m (\partial_{x'} - \partial_x)^n g(t',x') f(t,x)\Big|_{\substack{x'=x \\ t'=t}}$$

$$= (-1)^{m+n} \left[D_t^m D_x^n (g,f)\right](t,x)\ .$$

3. Compute the Hirota derivatives $D_t^2(f,g)$ and $D_x^4(f,g)$, and verify that your expression for the latter is consistent with the result for $D_x^4(f,f)$ given in lectures.

$$D_t^2(f,g) = (\partial_t - \partial_{t'})^2 f(t,x) g(t',x')\Big| = (\partial_t^2 - 2\partial_t \partial_{t'} + \partial_{t'}^2) f(t,x) g(t',x')\Big|$$

$$= f_{tt}(t,x)\, g(t',x') - 2 f_t(t,x)\, g_{t'}(t,x) + f(t,x)\, g_{t't'}(t',x')\Big|$$

$$= f_{tt}\, g - 2 f_t\, g_t + f g_{tt}\ .$$

$$D_x^4(f,g) = (\partial_x - \partial_{x'})^4 f(t,x) g(t',x')\Big| = (\partial_x^4 - 4\partial_x^3 \partial_x' + 6\partial_x^2 \partial_{x'}^2 - 4\partial_x \partial_{x'}^3 + \partial_{x'}^4) f(t,x) \cdot g(t',x')\Big|$$

$$= \underline{f_{xxxx}\, g} - 4 \underline{f_{xxx}\, g_x} + 6 f_{xx}\, g_{xx} - 4 \underline{f_x\, g_{xxx}} + \underline{f g_{xxxx}}\ .$$

$$D_x^4(f,f) = 2\left(f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx}^2\right) \qquad \checkmark\ .$$

Ex 37

Define a "non-Hirota" bilinear differential operator $\tilde{D}_t^m \tilde{D}_x^n$ by

$$\tilde{D}_t^m \tilde{D}_x^n(f,g) = \left(\frac{\partial}{\partial t} \underset{\uparrow\atop -}{+} \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} \underset{\uparrow\atop -}{+} \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Big|_{\substack{x'=x\\ t'=t}} \quad \text{for Hirota} \tag{6.6}$$

(note the plus signs!).

1. Compute $\tilde{D}_x(f,g)$ and $\tilde{D}_t(f,g)$, verifying that in both cases the answer is given by the corresponding partial derivative ∂_x or ∂_t of the product $f(x,t)g(x,t)$.
2. How does this result generalise for arbitrary non-Hirota differential operators (6.6)? Prove your claim.
3. Compare your answer with the Hirota operators defined above.

1. $[\tilde{D}_x(f,g)](t,x) = (\partial_x + \underset{-}{\partial_{x'}})\, f(t,x)\, g(t',x')\Big| = f_x\, g + f\, g_x = \partial_x(fg)$.

and similarly for $\tilde{D}_t$.

2. $(\underset{-}{\partial_t} + \underset{-}{\partial_{t'}})^m(\underset{-}{\partial_x} + \partial_{x'})^n f(t,x)\, g(t',x') = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}\left(\partial_t^k \partial_x^\ell f(t,x)\right)\left(\partial_{t'}^{m-k}\partial_{x'}^{n-\ell}\, g(t',x')\right)$

$$\Rightarrow \tilde{D}_t^m \tilde{D}_x^n(f,g) = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}(\partial_t^k\partial_x^\ell f)(\partial_t^{m-k}\partial_x^{n-\ell} g).$$

$\partial_t^m \partial_x^n\, f\cdot g$ $\leftarrow$ (Each derivative acts either on f or on g)

$$= \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}(\partial_t^k\partial_x^\ell f)(\partial_t^{m-k}\partial_x^{n-\ell} g) \qquad \text{same as above.}$$

$\binom{m}{k}$: # ways of splitting m ∂_t derivatives into k derivatives acting on f and $m-k$ derivatives acting on g

$\binom{n}{\ell}$: (same)

3. $$D_t^m D_x^n(f,g) = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}\boxed{(-1)^{m+n-k-\ell}}(\partial_t^k\partial_x^\ell f)(\partial_t^{m-k}\partial_x^{n-\ell} g)$$

Alternating sign $(-1)^{\#\text{ derivatives acting on } g}$

Ex 38

1. If $\theta_i = a_i x + b_i t + c_i$, prove that

$$D_t D_x(e^{\theta_1}, e^{\theta_2}) = (b_1 - b_2)(a_1 - a_2)e^{\theta_1+\theta_2} \ . \tag{6.7}$$

2. Prove the corresponding result for $D_t^m D_x^n(e^{\theta_1}, e^{\theta_2})$, as quoted in lectures.

1. $$D_t D_x (e^{\theta_1}, e^{\theta_2}) = (\partial_t - \partial_{t'})(\partial_x - \partial_{x'})\, e^{a_1x+b_1t+c_1 + a_2x'+b_2t'+c_2}\Big|$$
$$= (b_1 - b_2)(a_1 - a_2)\, e^{a_1x+b_1t+c_1+a_2x'+b_2t'+c_2}\Big|$$
$$= (b_1-b_2)(a_1-a_2)\, e^{\theta_1+\theta_2}$$

2. $$D_t^m D_x^n (e^{\theta_1}, e^{\theta_2}) = (b_1-b_2)^m (a_1-a_2)^n e^{\theta_1+\theta_2}$$

. . .