1. Numerical results seen in the lectures suggest that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

has an exact solution of the form

$$u(x,t) = \frac{2}{\cosh^2(x - vt)}$$

for some constant velocity v. Verify this by direct substitution into the KdV equation and determine the value of v.

Solution Direct substitution should confirm that the KdV equation is solved by the function given in the question if (and only if) $v = 4$.

2. (a) Show that if $u(x,t) = v(x,t)$ solves the KdV equation then so does $Av(Bx, Ct)$, provided that the constants B and C are related to A in a specific way (which you should determine).

   (b) Apply this transformation to the basic KdV solution found in problem 1 to construct a one-parameter family of one-soliton solutions of the KdV equation.

   (c) Find a formula relating the velocities to the heights for solitons in this one-parameter family. How does the width of a soliton in this family change if its velocity is rescaled by a factor of 4?

Solution

(a) Let $u(x,t) = Av(X,T)$, where $X = Bx$ and $T = Ct$. By the chain rule

$$u_t(x,t) = ACv_T(X,T)\,, \quad u_x(x,t) = ABv_X(X,T)\,, \quad u_{xxx}(x,t) = AB^3 v_{XXX}(X,T)\,,$$

so the left-hand side of the KdV equation for $u(x,t) = Av(Bx, Ct)$ becomes

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= ACv_T(X,T) + 6A^2Bv(X,T)v_X(X,T) + AB^3 v_{XXX}(X,T) \\ &= AC\left[v_T(X,T) + \frac{AB}{C}\cdot 6v(X,T)v_X(X,T) + \frac{B^3}{C}\cdot v_{XXX}(X,T)\right]\,. \end{aligned}$$

If $AB/C = B^3/C = 1$, this becomes

$$= AC\left[v_T(X,T) + 6v(X,T)v_X(X,T) + v_{XXX}(X,T)\right]\,,$$

which vanishes because $v(X,T)$ solves the KdV equation $v_T + 6vv_X + v_{XXX} = 0$ in its variables X and T by assumption. The two algebraic equations among the parameters are solved by $A = B^2$ and $C = B^3$. So

$$u(x,t) = v(x,t) \text{ solves KdV} \quad \Longrightarrow \quad u(x,t) = B^2 v(Bx, B^3 t) \text{ solves KdV } \forall B \in \mathbb{R}\,.$$

(b) Here our initial solution is $v(x,t) = 2\,\mathrm{sech}^2(x - 4t)$. By the above logic,

$$u(x,t) = 2B^2 \cdot \mathrm{sech}^2\left(B(x - 4B^2 t)\right)$$

is a family of solutions of the KdV equation labelled by a single real parameter B, which we can take to be positive wlog since sech^2 is an even function.

(c) The height is the maximum of u, and the velocity v is read off from the dependence on x, t through the single linear combination $x - vt$. We find that height $= 2B^2$ and velocity $= 4B^2$, so velocity $= 2 \times$ height.
The width is a measure of how much the lump is concentrated in space. Since the dependence on the spatial coordinate x is only through Bx, we deduce that width $\sim 1/B$.[1] The precise proportionality factor depends on the precise definition of width that you might choose, but regardless of that choice

$$\text{velocity} \mapsto 4\times\text{velocity} \equiv B^2\times\text{velocity} \quad \Longrightarrow \quad \text{width} \mapsto \frac{1}{B}\times\text{width} = \frac{1}{2}\times\text{width}\,.$$

3. Show that if $u(x,t)$ solves the KdV equation and ϵ is a constant, then $v(x,t) := \frac{1}{\epsilon}u(x,t)$ solves the rescaled KdV equation

$$v_t + 6\epsilon v v_x + v_{xxx} = 0\,,$$

while $w(x,t) := \epsilon u(x, \epsilon t)$ solves the differently-rescaled KdV equation

$$w_t + 6ww_x + \epsilon w_{xxx} = 0\,.$$

Solution This works much as the last question. Note that taking the limits $\epsilon \to 0$ in the two equations shows that the dispersive and dispersionless KdV equations discussed in the first lecture are somehow 'hidden' inside the usual KdV equation. However (especially in the second case) the limit is quite subtle...

4. Consider a pair of solitons with velocities m and n in the ball and box model, with $m > n$ and the faster soliton to the left of the slower one, with separation $l \geq n$ (*i.e.* there are $l \geq n$ empty boxes between the two solitons). Evolve various such initial conditions forward in time using the ball and box rule, for different values of m, n and l. start the solitons at least m boxes apart, so that interactions don't start until after the first time-step. Prove that the system always evolves into an oppositely-ordered pair of the same two solitons, and find a general formula for the phase shifts[2] of the solitons in terms of m and n.

[Optional:] What can go wrong if $l < n$? **[Hint:** Evolve the system backwards...]

Solution Rather than discussing examples, I will give a proof and calculate the phase shifts in full generality, but you will get credit for providing examples as long as they are correct.
The velocity m soliton has length m and (by definition) moves by m boxes in one unit of time when it is far away from other solitons. Likewise, the length n soliton moves by n boxes in one unit of time when it is far away from other solitons. Therefore, when the faster length m soliton is far enough behind the slower length n soliton, the separation

[1] If you thought that width $\sim B$ rather than $1/B$, pick your favourite localised function $f(x)$, plot $f(x)$ and $f(2x)$ and compare: is the width of $f(2x)$ double or half the width of $f(x)$?

[2] The phase shift of a soliton is defined to be the shift of its position, at a time in the far future, relative to the position it would have had at the same time if the other soliton hadn't been there.

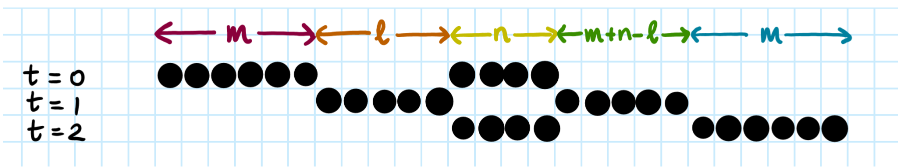
decreases as follows in one unit of time (the subscript denotes the value of the discrete time coordinate):

$$l_t = l \quad \mapsto \quad l_{t+1} = l - m + n \, .$$

We can iterate this process until the separation l reaches the range

$$n \leq l < m \, ,$$

at a time that I will label as $t = 0$ in the following (this can be achieved by a shift of the time coordinate). Let's now evolve the system forward from $t = 0$, using the above inequality and taking into account that boxes which are full at time t are empty at time $t + 1$:



At $t = 2$ the slower length n soliton is more than n boxes behind the faster length m soliton, so the collision is over and we don't need to evolve the system any further to calculate the phase shifts. At $t = 2$ the faster length m soliton has moved by $m + l + n + (m + n - l) = 2(m + n)$ boxes compared to where it was at $t = 0$. In the absence of the slower soliton, it would have moved by $2m$ boxes. The slower length n soliton has not moved at $t = 2$ compared to where it was at $t = 0$. In the absence of the faster soliton, it would have moved by $2n$ boxes. The differences between the positions of the solitons after the collision and the positions the solitons would have had before the collision are the phase shifts

$$\begin{aligned} (\text{phase shift})_{\text{faster}} &= 2(m + n) - 2m = 2n \, , \\ (\text{phase shift})_{\text{slower}} &= 0 - 2n = -2n \, . \end{aligned}$$

5. In the two-colour (blue and red) ball and box model, we'll call a row of n consecutive balls a soliton if it keeps its form over time, so that after each time-step its only change is a possible (fixed) translation. There's no need for both colours to be represented, so a row of n blue balls, or a row of n red balls, is also a potential soliton. How many solitons of length n are there? What are their speeds?

   Solution A first thing to notice is that a sequence of any number $a \geq 0$ of blue balls, followed by any number $b \geq 0$ red balls, will be a soliton: the blue balls move first, taking up positions to the right of the red ones, and then the red balls move and reproduce the same pattern as before, $a + b$ places to the right. If the length, $a + b$, is n, this gives $n + 1$ possibilities (since a can be equal to $0, 1, \dots n$, with b then equal to $n - a$). All of these move with speed n. If n is even then there is one further possibility: consider a configuration where the first $n/2$ balls are red, followed by $n/2$ blue balls. In one time-step, first the blue balls all move $n/2$ places to the right, leaving a gap of length

$n/2$ into which the red balls move. This soliton has speed $n/2$. This suggests that the full answer is that the number of solitons is equal to $n + 1$ when n is odd (all with speed n), and $n + 2$ when n is even ($n + 1$ of them with speed n, and 1 with speed $n/2$).

This turns out to be correct, though to prove it needs a little thought. Consider first a possible soliton that starts with $a > 0$ blue balls on the far left. To be something new it must continue with some number $b > 0$ of red balls, and then $c > 0$ blue ones. After one time step, the blue balls will have moved away to the right, and a red ball will have moved into the space previously occupied by the leftmost of the block of c blue balls. This is now the leftmost ball; so after one time step the leftmost ball has changed from blue to red, and so this is not a soliton. Next, suppose we start with $a > 0$ red balls and then $b > 0$ blue balls. If $b > a$ then after the blue balls have moved there will not be enough red balls to fill the space created and so the configuration after one time-step will have a gap; similarly there will be a gap after one time-step if $b < a$. Hence $b = a$ and again considering that no gaps can form after one time-step, there can be no further red balls after these two blocks. This suffices to show that there are no solitons beyond those already listed in the previous paragraph.

6. The ball and box model can be further generalised to the M-colour ball and box model. The balls now come in M colours, $1, 2, \dots, M$, and the time-evolution rule is generalised to say that first all balls of colour 1 are moved, then all of colour 2, and so on, with a single time-step being completed once all balls of all colours have been moved. How many solitons of length n are there in this model? Again, there is no need for every colour to be present in a given soliton. You might start by classifying the 'top-speed' solitons of length n, that is, those that move at speed n.

   Solution This is harder! We can discuss it further informally, but the answer for speed n at least is relatively simple - a soliton must start with some (possibly-zero) number $n_1 \geq 0$ of balls of colour 1, then $n_2 \geq 0$ of colour 2, up to $n_n \geq 0$ of colour M, with $n_1 + n_2 + \dots n_n = n$. It's easy to see that this *is* a soliton, and a little harder to see that all speed-n solitons have this form. After that it is a problem in combinatorics to count the number of such solitons; the answer you should find is $\binom{n+M-1}{M-1}$.

7. Investigate the scattering of solitons in the two-colour ball and box model. You should find that the lengths of top-speed solitons are preserved under collisions, but their forms can change. Try to formulate a general rule for this behaviour. Can you generalise it to the M-colour model?

   Solution Left for the enthusiastic student!

8. (a) Express d'Alembert's general solution of the wave equation $u_{tt} - u_{xx} = 0$ in terms of the initial conditions $u(x, 0) = p(x)$ and $u_t(x, 0) = q(x)$.

   (b) Find a relation between $p(x)$ and $q(x)$ which produces a single wave travelling to the right.

Solution

(a) D'Alembert's general solution is

$$u(x,t) = f(x-t) + g(x+t)$$

for arbitrary functions f and g. To match the initial conditions we need

$$\text{(A) } u(x,0) = f(x) + g(x) = p(x)\,, \quad \text{(B) } u_t(x,0) = -f'(x) + g'(x) = q(x)\,.$$

Equation (B) implies

$$-f(x) + g(x) = \int_{x_0}^{x} q(s)\,ds$$

for some constant x_0. Hence $g(x) = f(x) + \int_{x_0}^{x} q(s)\,ds$ and substituting into (A),

$$2f(x) + \int_{x_0}^{x} q(s)\,ds = p(x) \quad \Rightarrow \quad f(x) = \tfrac{1}{2}p(x) - \tfrac{1}{2}\int_{x_0}^{x} q(s)\,ds$$

and likewise $g(x) = \frac{1}{2}p(x) + \frac{1}{2}\int_{x_0}^{x} q(s)\,ds$. Adding up,

$$\begin{aligned} u(x,t) &= f(x-t) + g(x+t) \\ &= \tfrac{1}{2}(p(x-t) + p(x+t)) - \tfrac{1}{2}\int_{x_0}^{x-t} q(s)\,ds + \tfrac{1}{2}\int_{x_0}^{x+t} q(s)\,ds \\ &= \tfrac{1}{2}(p(x-t) + p(x+t)) + \tfrac{1}{2}\int_{x-t}^{x+t} q(s)\,ds\,. \end{aligned}$$

Note that $u(x,t)$ only depends on the initial data via $p(x-t)$, $p(x+t)$, and the values of $q(s)$ for $x-t \le s \le x+t$. The interval $[x-t, x+t]$ is sometimes called the *domain of dependence* of u at (x,t).

(b) For there to be a single wave travelling to the right, we need g to be a constant. Differentiating the formula found above for g, this requires $p' + q = 0$. (As a check on this formula, it is easy to see it holds at $t = 0$ for $u(x,t) = f(x-t)$.)

9. The wave profile

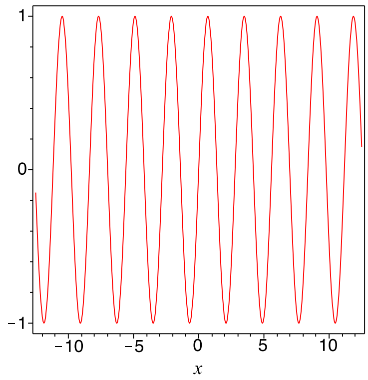
$$\phi(x,t) = \cos(k_1 x - \omega(k_1)t) + \cos(k_2 x - \omega(k_2)t)$$

is a superposition of two plane waves. Rewrite ϕ as a product of cosines, and use this to sketch the wave profile when $|k_1 - k_2| \ll |k_1|$. Find the velocity at which the envelope of the wave profile moves (the **group velocity**), again for $k_1 \approx k_2$; in the limit $k_1 \to k_2$ verify that this reduces to $d\omega/dk$, consistent with the general result obtained in lectures.

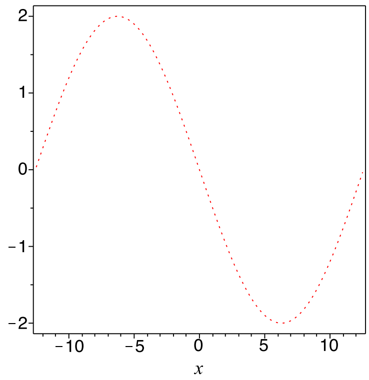
Solution This question gives an alternative insight into the formula for group velocity. Taking things bit-by-bit, Using $\cos a + \cos b = 2\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)$, the expression for $\phi(x,t)$ can be rewritten as

$$\phi(x,t) = 2\cos\left(\tfrac{k_1+k_2}{2}x - \tfrac{\omega_1(k_1)+\omega_2(k_2)}{2}t\right)\cos\left(\tfrac{k_1-k_2}{2}x - \tfrac{\omega_1(k_1)-\omega_2(k_2)}{2}t\right)$$

For $k_1 \approx k_2$, the factor involving $\frac{k_1+k_2}{2}$ has a much higher wavenumber, and hence shorter wavelength, than the one involving $\frac{k_1-k_2}{2}$. The first one corresponds to the carrier wave, and the second to the envelope. Taken separately, say at $t = 0$, they look like

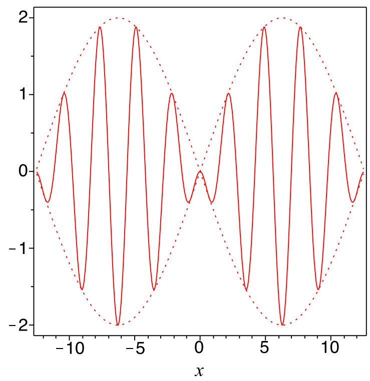


Short wavelength carrier wave at t=0, cos $\left(\frac{k_1+k_2}{2}\, x\right)$



Long wavelength envelope at t=0, 2 cos $\left(\frac{k_1-k_2}{2}\, x\right)$

and their product is



The full wave

The envelope function at time t is $2\cos(k_{\text{envelope}}x - \omega_{\text{envelope}}t)$ where $k_{\text{envelope}} = \frac{k_1-k_2}{2}$ and $\omega_{\text{envelope}} = \frac{\omega(k_1)-\omega(k_2)}{2}$. Its velocity is $v_{\text{envelope}} = \frac{\omega_{\text{envelope}}}{k_{\text{envelope}}} = \frac{\omega(k_1)-\omega(k_2)}{k_1-k_2}$. For $k_1 \to k_2$, $v_{\text{envelope}} \to \lim_{k_1\to k_2}\left(\frac{\omega_{\text{envelope}}}{k_{\text{envelope}}}\right) = \omega'(k_1) = \omega'(k_2)$, which, as expected, is equal to the group velocity.

10. (a) Completing the square, derive the formula

$$\int_{-\infty}^{+\infty} dk\, e^{-A(k-\bar{k})^2} e^{ikB} = \sqrt{\frac{\pi}{A}}\, e^{i\bar{k}B} e^{-B^2/(4A)}\ .$$

(You can quote the result $\int_{-\infty}^{+\infty} dk\, e^{-Ak^2} = \sqrt{\pi/A}$ for $A > 0$.)

(b) For the Gaussian wavepacket (where Re denotes the real part)

$$u(x,t) = \text{Re}\int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2} e^{i(kx-\omega(k)t)}\ ,$$

expand $\omega(k)$ to second order in $k - \bar{k}$, and then use the result of part (a) to derive a better approximation for $u(x,t)$ than that obtained in lectures.

(c) Given that a function of the form $e^{-(x-x_0)^2/C}$ describes a profile centred at x_0 with width^{-2} equal to the real part of C^{-1}, show that the result of part (b) is a wave profile moving at velocity $\omega'(\bar{k})$, with width^2 increasing with time as $4a^2 + \omega''(\bar{k})^2 t^2/a^2$. (Hence, for $\omega'' \neq 0$, the wave disperses.)

Solution Here we will approximate the time-dependence of the Gaussian wavepacket to one higher order than was done in lectures, to show that, so long as the dispersion relation is nontrivial, the wavepacket *does* disperse, or spread out, as time goes by. This means that it can't be a soliton – it fails property (2) from section 1.1!

(a)

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-A^2(k-\bar{k})^2} e^{ikB} dk &= e^{i\bar{k}B} \int_{-\infty}^{+\infty} e^{-A^2(k-\bar{k})^2} e^{iB(k-\bar{k})} dk \\
&= e^{i\bar{k}B} \int_{-\infty}^{+\infty} e^{-A^2k^2} e^{iBk} dk \\
&= e^{i\bar{k}B} \int_{-\infty}^{+\infty} e^{-A^2(k-\frac{iB}{2A^2})^2 - \frac{B^2}{4A^2}} dk \\
&= \frac{\sqrt{\pi}}{A} e^{i\bar{k}B} e^{-\frac{B^2}{4A^2}}
\end{aligned}$$

(b)

$$u(x,t) = \int_{-\infty}^{+\infty} e^{-a^2(k-\bar{k})^2} e^{i(kx-\omega(k)t)} dk$$

For $k \approx \bar{k}$,

$$\omega(k) = \omega(\bar{k}) + (k-\bar{k})\omega'(\bar{k}) + \frac{(k-\bar{k})^2}{2}\omega''(\bar{k}) + \ldots$$

We'll only keep these terms – given the damped nature of $e^{-a^2(k-\bar{k})^2}$ away from $k = \bar{k}$, this will be a reasonable approximation. Thus the improved approximation is

$$\begin{aligned}
u(x,t) &\approx \int_{-\infty}^{+\infty} e^{-a^2(k-\bar{k})^2} e^{i(kx-\omega(\bar{k})t-(k-\bar{k})\omega'(\bar{k})t+\frac{(k-\bar{k})^2}{2}\omega''(\bar{k}))t} dk \\
&= e^{i\omega(\bar{k})t+i\bar{k}\omega'(\bar{k})t} \int_{-\infty}^{\infty} e^{-(a^2+\frac{i\omega''(\bar{k})t}{2})(k-\bar{k})^2} e^{ik(x-\omega'(\bar{k})t)} dk \\
&= e^{i\omega(\bar{k})t+i\bar{k}\omega'(\bar{k})t} \, e^{i\bar{k}(x-\omega'(\bar{k})t)} \frac{\sqrt{\pi}}{\sqrt{(a^2+\frac{i\omega'(\bar{k})t}{2})}} e^{-\frac{(x-\omega'(\bar{k})t)^2}{4(a^2+\frac{i\omega''(\bar{k})t}{2})}} \\
&= \sqrt{\frac{2\pi}{2a^2+i\omega''(\bar{k})t}} \, e^{i(\bar{k}x-\omega(\bar{k})t)} \, e^{-\frac{(x-\omega'(\bar{k})t)^2}{4(a^2+\frac{i\omega''(\bar{k})t}{2})}} .
\end{aligned}$$

(c) The wave envelope found in (b) is of the form $e^{-(x-x_0)^2/C}$ with

i. $x_0 = \omega'(\bar{k})t$ and

ii. $C^{-1} = 4^{-1}(a^2 + \frac{i\omega''(\bar{k})t}{2})^{-1} = \frac{4^{-1}(a^2 - \frac{i\omega'' t}{2})}{a^4 + (\frac{\omega'' t}{2})^2}$

Hence the velocity of the envelope is $\frac{dx_0(t)}{dt} = \omega'(\bar{k})$ and the width squared is $\frac{1}{Re(C^{-1})} = 4a^2 + \frac{\omega''(\bar{k})^2 t^2}{a^2}$. So for large t the width grows at a rate which is proportional to $|d^2\omega(\bar{k})/dk^2|$.

11. Find the dispersion relation and the phase and group velocities for:

(a) $u_t + u_x + \alpha u_{xxx} = 0\,;$

(b) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}\,.$

Solution

(a) Substitute in a plane wave $u(x,t) = e^{i(kx-\omega t)}$ to get the algebraic equation $-i\omega + ik - i\alpha k^3 = 0$. So the dispersion relation is

$$\omega = \omega(k) = k - \alpha k^3 = k(1 - \alpha k^2)\;,$$

and the phase and group velocities are

$$\begin{aligned} \text{Phase velocity :} \qquad c(k) \;&= \frac{\omega(k)}{k} = 1 - \alpha k^2 \\ \text{Group velocity :} \qquad c_g(k) &= \omega'(k) = 1 - 3\alpha k^2\;. \end{aligned}$$

(b) Substitute in a plane wave $u(x,t) = e^{i(kx-\omega t)}$ to get the algebraic equation $-\omega^2 + \alpha^2 k^2 = \beta^2 k^2 \omega^2$. So the dispersion relation is[3]

$$\omega = \omega(k) = \pm\frac{\alpha k}{(1+\beta^2 k^2)^{1/2}}\;,$$

and the phase and group velocities are

$$\begin{aligned} \text{Phase velocity :} \qquad c(k) \;&= \frac{\omega(k)}{k} = \pm\frac{\alpha}{(1+\beta^2k^2)^{1/2}} \\ \text{Group velocity :} \qquad c_g(k) &= \omega'(k) = \pm\alpha\left[\frac{1}{(1+\beta^2k^2)^{1/2}} - \frac{k}{2}\frac{2\beta^2 k}{(1+\beta^2k^2)^{3/2}}\right] \\ &= \pm\frac{\alpha}{(1+\beta^2k^2)^{3/2}}\;. \end{aligned}$$

[3]We might restrict to the positive solution for the dispersion relation as in the lecture notes, since the negative solution is obtained by taking the complex conjugate of the plane wave and reversing the sign of k. I won't do it in this solution, but your marks will not be affected by the sign you picked whether you wrote $+$ or $\pm$.

12. For which values of n does the equation

$$u_t + u_x + u_{xxx} + \frac{\partial^n u}{\partial x^n} = 0$$

admit "physical" dissipation? (A wave is said to have physical dissipation if the amplitude of plane waves decreases with time.)

Solution Substituting in, $\omega(k)$ must satisfy

$$\begin{aligned} -i\omega + ik + (ik)^3 + (ik)^n &= 0 \\ \Rightarrow \quad \omega &= k - k^3 + i^{n-1}k^n \end{aligned}$$

So the solution is $u = e^{i(kx-(k-k^3)t)}e^{-i^n k^n t}$. There are dissipative solutions if and only if i^n is real, that is if and only if n is even. These have physical dissipation if and only if $i^n = 1$, so the condition is $n = 0$ modulo 4.

13. Find (if possible) real non-singular travelling wave solutions of the following equations, satisfying the given boundary conditions:

(a) Modified KdV (mKdV) equation:

$$\begin{aligned} &u_t + 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\ . \end{aligned}$$

(b) 'Wrong sign' mKdV equation:

$$\begin{aligned} &u_t - 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\ . \end{aligned}$$

(c) ϕ^4 theory:

$$\begin{aligned} &u_{tt} - u_{xx} + 2u(u^2-1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to -1 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to +1 \ \text{ as } x \to +\infty\ . \end{aligned}$$

(d) ϕ^6 theory:

$$\begin{aligned} &u_{tt} - u_{xx} + u(u^2-1)(3u^2-1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to 0 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to 1 \ \text{ as } x \to +\infty\ . \end{aligned}$$

(e) Burgers equation:

$$\begin{aligned} &u_t + uu_x - u_{xx} = 0 \\ &u \to u_0,\ u_x \to 0 \ \text{ as } x \to -\infty \\ &u \to u_1,\ u_x \to 0 \ \text{ as } x \to +\infty\ , \end{aligned}$$

where u_0 and u_1 are real constants with $u_0 > u_1 > 0$.
[Hint: Start by showing that the boundary conditions relate the velocity v of the travelling wave to the sum of the constants u_0 and u_1.]

Solution

(a) The equation is:

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

We look for $u(x,t) = f(x - vt)$. Integrating once,

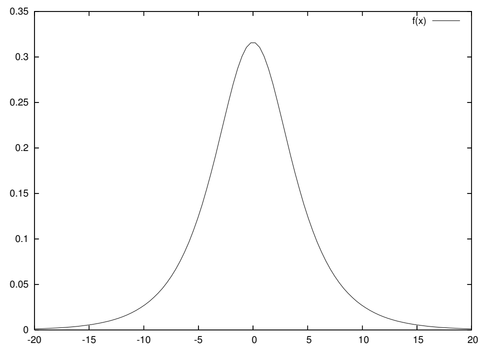
$$-vf + 2f^3 + f'' = A\,;$$

using the given boundary conditions, $A = 0$.
Multiply by f' and integrate again:

$$-\frac{vf^2}{2} + \frac{f^4}{2} + \frac{1}{2}f'^2 = B\,.$$

Using boundary conditions again, $B = 0$. Hence

$$\begin{aligned} f' &= \pm\sqrt{vf^2 - f^4} \\ \Rightarrow\ x - vt &= \pm\int \frac{df}{f\sqrt{v - f^2}} \\ &= \pm\frac{1}{\sqrt{v}}\text{sech}^{-1}\frac{f}{\sqrt{v}} + x_0 \\ \Rightarrow\ u(x,t) &= \pm\sqrt{v}\,\text{sech}(\sqrt{v}(x - x_0 - vt)) \end{aligned}$$



A travelling wave for the mKdV equation

(b) If the sign in the non-linear term is changed, the equation turns into the 'alternative' mKdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0$$

and so

$$\begin{aligned} x - vt &= \pm\int \frac{df}{f\sqrt{v + f^2}} \\ \Rightarrow\ u(x,t) &= \pm\sqrt{v}\,\text{csch}(\sqrt{v}(x - x_0 - vt)) \end{aligned}$$

The solution is singular at $x = x_0 + vt$, and so the alternative mKdV equation has no nonsingular solitary wave solutions of this type (that is, with $u(\pm\infty) = 0$).

(c) As before, we assume that the solution is of the form

$$u(x,t) = f(x - vt)$$

for some real v. The equation of motion becomes

$$2f(f^2-1) - (1-v^2)f'' = 0\,.$$

Multiplying by f' and integrating yields

$$\frac{1}{2}(1-v^2)(f')^2 = \frac{1}{2}f^4 - f^2 + A$$

where A is a constant. Taking the boundary conditions into account, remembering that the limiting value of f at $\pm\infty$ is ± 1, we have $A = \frac{1}{2}$ and so the right-hand side is equal to $\frac{1}{2}(f^2-1)^2$. Hence, with $\gamma = \frac{1}{\sqrt{1-v^2}}$,

$$f' = \pm\gamma(f^2-1)$$

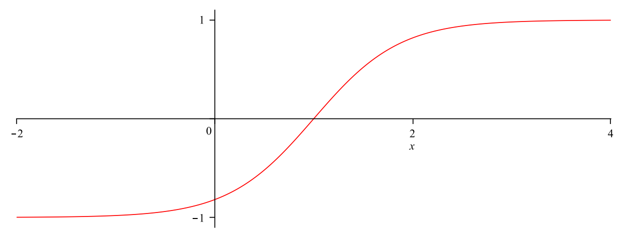
Integrating this equation,

$$\mathrm{arctanh}(f) = \pm\gamma(x - x_0 - vt)$$

where x_0 is the constant of integration. So travelling wave is

$$u(x,t) = \pm\tanh(\gamma(x - x_0 - vt))\,.$$

If the sign is positive, we have a kink; if negative, an anti-kink. To match with the boundary conditions given in the question, the positive sign should be chosen.



A ϕ^4 kink at $t = 0$, with $v = 0.25$ and $x_0 = 1$

(d) The ϕ^6 theory: this is left as an exercise for the keen student! (It's good practice.)

(e) Burger's equation is $u_t + uu_x - u_{xx} = 0$, so

$$\begin{aligned} u(x,t) = f(x-vt) \quad &\Rightarrow \quad -vf' + ff' - f'' = 0 \\ &\Rightarrow \quad -vf + \tfrac{1}{2}f^2 - f' = A \end{aligned}$$

for some constant A. The boundary conditions at $x = -\infty$ imply $A = -vu_0 + \frac{1}{2}u_0^2$, while at $x = +\infty$ they imply $-vu_1 + \frac{1}{2}u_1^2 = A$. Equating the two expressions for A,

$$-vu_0 + \tfrac{1}{2}u_0^2 = -vu_1 + \tfrac{1}{2}u_1^2\,.$$

Solving for v,

$$v = \tfrac{1}{2}(u_0 + u_1)$$

as suggested by the hint in the question.
The ODE to be solved is therefore

$$f' = \tfrac{1}{2}\big(f^2 - (u_0 + u_1)f + u_0 u_1\big) = \tfrac{1}{2}(f - u_0)(f - u_1)$$

This is separable, and so (with x_0 the constant of integration)

$$\tfrac{1}{2}(x - x_0 - vt) = \int \frac{df}{(f - u_0)(f - u_1)}$$

Next use partial fractions, taking care that we expect the nonsingular solution to interpolate continuously between $f = u_0$ and $f = u_1$:

$$\begin{aligned} \int \frac{df}{(f-u_0)(f-u_1)} &= \int \frac{df}{(u_0 - u_1)} \left(\frac{-1}{(u_0 - f)} - \frac{1}{(f - u_1)} \right) \\ &= \frac{1}{(u_0 - u_1)} \left(\log(u_0 - f) - \log(f - u_1) \right) . \end{aligned}$$

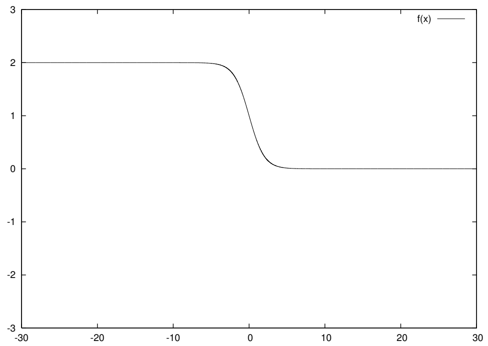
Hence

$$\tfrac{1}{2}(u_0 - u_1)(x - x_0 - vt) = \log\left(\frac{u_0 - f}{f - u_1}\right) \;\Rightarrow\; \frac{u_0 - f}{f - u_1} = e^{\frac{1}{2}(u_0 - u_1)(x - x_0 - vt)}$$

There's a tricky point here: it's important to choose the right branches of the logs, so that for the range of f of interest we are always taking logs of *positive* numbers. Solving for f and using $u(x,t) = f(x - vt)$ we finally get to

$$u(x,t) = f(x - vt) = \frac{u_0 + u_1 e^{\frac{1}{2}(u_0 - u_1)(x - x_0 - vt)}}{1 + e^{\frac{1}{2}(u_0 - u_1)(x - x_0 - vt)}}$$

with $v = \frac{1}{2}(u_0 + u_1)$. An example solution for $u_1 = 0$ is plotted below. A 'wrong' choice for the branches of the log would have given $1 - e^{\frac{1}{2}(u_0 - u_1)(x - x_0 - vt)}$ in the denominator, and the solution would have been singular at the point $x = x_0 + vt$.



A travelling wave for Burger's equation

14. Using the analogy with the classical mechanics of a point particle moving in one spatial dimension, determine the qualitative behaviour of travelling wave solutions of the KdV equation on a circle, for which the integration constants A and B are non-zero.

    Solution This is left for you to think about, but as a start, you might consider the two possible ways a ball rolling in a corrigated landscape can behave, depending on whether it has enough energy to get over the ridges or not.

15. This exercise involves the infinite chain of identical coupled pendulums of section 3.3, whose equations of motion reduce to the sine-Gordon equation in the continuum limit $a \to 0$. We will simplify expression by setting $g = L = \frac{M}{a} = 1$. Let $\theta_n(t)$ be the angle to the vertical of the n-th pendulum ($n \in \mathbb{Z}$), which is hung at the position $x = na$ along the chain, at time t. The configuration of the system at time t is then specified by the collection of angles $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

    (a) Starting from the force (note: m is a dummy variable)

    $$F_n(\{\theta_m\}) = -a\sin\theta_n + \frac{1}{a}(\theta_{n+1} - \theta_n) + \frac{1}{a}(\theta_{n-1} - \theta_n)$$

    acting on the n-th pendulum, deduce the potential energy

    $$V(\{\theta_m\}) = \sum_{n=-\infty}^{+\infty} (\cdots)$$

    such that $F_n = -\frac{\partial V}{\partial \theta_n}$ for all $n \in \mathbb{Z}$, and fix the integration constant by requiring that the potential energy be zero when all pendulums point down: $V(\{0\}) = 0$.

    (b) Show that in the continuum limit $a \to 0$, the potential energy computed above becomes

    $$V = \int_{-\infty}^{+\infty} dx \, \left[(1 - \cos\theta) + \frac{1}{2}\theta_x^2\right] ,$$

    and the kinetic energy

    $$T(\{\theta_m\}) = \frac{a}{2}\sum_{n=-\infty}^{+\infty} \dot{\theta}_n^2$$

    becomes

    $$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2}\theta_t^2 ,$$

    where the function $\theta(x,t)$ is the continuum limit of $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

    [**Hint**: in the continuum limit, $a \sum\limits_{n=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty} dx$.]

Solution

(a) We can write $F_n(\{\theta_m\}) = F_n^{\text{grav}} + F_n^{\text{twist}}$, where $F_n^{\text{grav}} = -a\sin\theta_n$ and $F_n^{\text{twist}} = \frac{1}{a}(\theta_{n+1}-\theta_n) + \frac{1}{a}(\theta_{n-1}-\theta_n)$. Imposing the relations

$$F_n^{\text{grav}} = -\frac{\partial}{\partial\theta_n}V^{\text{grav}}\ , \quad F_n^{\text{twist}} = -\frac{\partial}{\partial\theta_n}V^{\text{twist}}$$

and integrating gives us

$$V^{\text{grav}}(\{\theta_m\}) = a\sum_{m=-\infty}^{+\infty}(1-\cos\theta_m)\ , \quad V^{\text{twist}}(\{\theta_m\}) = \frac{1}{2a}\sum_{m=-\infty}^{+\infty}(\theta_{m+1}-\theta_m)^2\ ,$$

where the constants of integration were chosen so that $V^{\text{grav}} = 0$ when all the pendulums are pointing down. (You should check that this works!) Hence

$$V(\{\theta_m\}) = a\sum_{m=-\infty}^{+\infty}(1-\cos\theta_m) + \frac{1}{2a}\sum_{m=-\infty}^{+\infty}(\theta_{m+1}-\theta_m)^2\ .$$

(b) In the continuum limit the set of functions $\{\theta_n(t)\}$ approximates a continuous function $\theta(x,t)$ such that

$$\theta_n(t) = \theta(x{=}na, t)\ .$$

By the definition of the derivative as a limit we have

$$\lim_{a\to 0}\left(\frac{\theta_{n+1}(t)-\theta_n(t)}{a}\right) = \theta_x(x{=}na, t)\ ,$$

and by the definition of the integral as the limit of a sum,

$$\lim_{a\to 0}\sum_{n=-\infty}^{\infty} af(\theta_n(t)) = \int_{-\infty}^{\infty} f(\theta(x,t))\,dx\ .$$

In this case

$$\begin{aligned} V &= \sum_{m=-\infty}^{+\infty} a\left[(1-\cos\theta_m) + \frac{1}{2}\left(\frac{\theta_{m+1}-\theta_m}{a}\right)^2\right] \\ &\to \int_{-\infty}^{\infty}\left[(1-\cos\theta(x,t)) + \tfrac{1}{2}\theta_x(x,t)^2\right]dx \end{aligned}$$

Likewise

$$\begin{aligned} T &= \sum_{m=-\infty}^{+\infty} a\frac{1}{2}\dot{\theta}_m^2 \\ &\to \int_{-\infty}^{\infty}\tfrac{1}{2}\theta_t(x,t)^2 dx \end{aligned}$$

as required.

16. A field $u(x,t)$ has kinetic energy T and potential energy V, where

$$T = \int_{-\infty}^{+\infty} dx\, \frac{1}{2} u_t^2 \,,$$
$$V = \int_{-\infty}^{+\infty} dx\, \left[\frac{1}{2}u_x^2 + \frac{\lambda}{2}(u^2-a^2)^2\right] \,,$$

and a and $\lambda > 0$ are (real) constants. (This is a version of the 'ϕ^4' theory, so named because the scalar potential is quartic, and the field u is usually called ϕ.) The equation of motion for u is

$$u_{tt} - u_{xx} + 2\lambda u(u^2-a^2) = 0\,.$$

(a) If u is to have finite energy, what boundary conditions must be imposed on u, u_x and u_t at $x = \pm\infty$?

(b) Find the general travelling-wave solutions to the equation of motion, consistent with the boundary conditions found in part (a). Compute the total energy $E = T + V$ for these solutions. For which velocity do the solutions have the lowest energy?

(c) One of the possible boundary conditions for part (a) implies that u is a kink, with $[u(x)]_{x=-\infty}^{x=+\infty} = 2a$. Use the Bogomol'nyi argument to show that the total energy $E = T+V$ of that configuration is bounded from below by $C\sqrt{\lambda}a^3$, where C is a constant that you should determine, and find the solution u which saturates this bound. Verify that this solution agrees with the lowest-energy solution of part (b).

Solution

(a) Since all terms in T and V are positive, in order to have a finite energy they must all tend to zero as $|x| \to +\infty$, so

$$(u_t)^2 \to 0 \implies u_t \to 0$$
$$(u_x)^2 \to 0 \implies u_x \to 0$$
$$(u^2-a^2)^2 \to 0 \implies u \to a \text{ or } -a \;\; \text{(independently at } -\infty \text{ and } +\infty)$$

(Strictly speaking they must also tend to their limiting values *fast enough* that all the integrals converge, but we won't need such a level of detail here.)

(b) The two options for boundary conditions leading to nontrivial travelling waves are

$$u(-\infty,t) = -a,\;\; u(+\infty,t) = +a \quad \to \quad u(x,t) = a\tanh\left[a\sqrt{\lambda}\gamma(x-x_0-vt)\right]$$
$$u(-\infty,t) = +a,\;\; u(+\infty,t) = -a \quad \to \quad u(x,t) = -a\tanh\left[a\sqrt{\lambda}\gamma(x-x_0-vt)\right]$$

where $\gamma = 1/\sqrt{1-v^2}$.
The working is as in problem 13 (c), taking into account the fact that $a,\ \lambda \neq 1$. (Either re-do from scratch, or else rescale $(u,x,t) \to (au, a\sqrt{\lambda}x, a\sqrt{\lambda}t)$.)

For these solutions, and using $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$ and $\tanh^2(x)-1 = -\operatorname{sech}^2(x)$,

$$(u_t)^2 = a^4\lambda\gamma^2 v^2 \operatorname{sech}^4\left[a\sqrt{\lambda}\gamma(x-x_0-vt)\right]$$
$$(u_x)^2 = a^4\lambda\gamma^2 \operatorname{sech}^4\left[a\sqrt{\lambda}\gamma(x-x_0-vt)\right]$$
$$\lambda(u^2-a^2)^2 = a^4\lambda \operatorname{sech}^4\left[a\sqrt{\lambda}\gamma(x-x_0-vt)\right] .$$

Hence

$$\begin{aligned} E &= \frac{1}{2}\left(a^4\lambda\gamma^2v^2 + a^4\lambda\gamma^2 + a^4\lambda\right)\int_{-\infty}^{\infty}\operatorname{sech}^4\left[a\sqrt{\lambda}\gamma(x-x_0-vt)\right]dx \\ &= \frac{a^4\lambda}{2}\left(\gamma^2(1+v^2)+1\right)\frac{1}{a\sqrt{\lambda}\gamma}\int_{-\infty}^{\infty}\operatorname{sech}^4(y)dy \\ &= \frac{a^3\sqrt{\lambda}}{2}\frac{2\gamma}{1-v^2}\frac{4}{3} = \frac{4}{3}a^3\sqrt{\lambda}\,\gamma\,. \end{aligned}$$

Since $\gamma = 1/\sqrt{1-v^2}$ the lowest-energy solutions are those with $v = 0$, and for them, $E = \frac{4}{3}a^3\sqrt{\lambda}$.
There are also the options to take $u(-\infty,t) = u(\infty,t) = -a$ or $u(-\infty,t) = u(\infty,t) = a$ which lead to the constant solutions $u(x,t) = -a$ and $u(x,t) = a$, which have zero energy, but I wouldn't insist on these being mentioned in your answers.

(c) For the total energy of a general configuration (not necessarily a travelling wave), we have:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty}\left(\tfrac{1}{2}(u_t)^2 + \tfrac{1}{2}(u_x)^2 + \tfrac{\lambda}{2}(u^2-a^2)^2\right)dx \\ &\geq \int_{-\infty}^{+\infty}\left(\tfrac{1}{2}(u_x)^2 + \tfrac{\lambda}{2}(u^2-a^2)^2\right)dx \\ &= \int_{-\infty}^{+\infty}\tfrac{1}{2}\left(u_x \pm \sqrt{\lambda}(u^2-a^2)\right)^2 \mp \sqrt{\lambda}(u^2-a^2)u_x\,dx \\ &= \int_{-\infty}^{+\infty}\tfrac{1}{2}\left(u_x \pm \sqrt{\lambda}(u^2-a^2)\right)^2 dx \mp \sqrt{\lambda}\left[\tfrac{1}{3}u^3 - a^2u\right]_{-\infty}^{\infty} \end{aligned}$$

where we used $(u^2-a^2)u_x = \frac{d}{dx}\left(\frac{1}{3}u^3 - a^2u\right)$ to get to the last line. Given the kink boundary conditions $-u(-\infty) = u(+\infty) = a$, this is

$$E \geq \int_{-\infty}^{+\infty}\tfrac{1}{2}\left(u_x \pm \sqrt{\lambda}(u^2-a^2)\right)^2 dx \pm \tfrac{4}{3}\sqrt{\lambda}a^3$$

which is actually *two* equations, one with the plus signs and one with the minus signs. If we take the version with the plus signs, using the fact that the integrand is non-negative, we immediately deduce that

$$E \geq \frac{4}{3}\sqrt{\lambda}a^3$$

which is of the desired form, with $C = \frac{4}{3}$. If we'd taken the minus signs we'd have gotten a true fact – namely that $E \geq -\frac{4}{3}\sqrt{\lambda}a^3$ – but since we already knew that $E \geq 0$, this wouldn't have told us anything new.

In order to saturate the bound, the integrand in the last inequality must vanish for all points x. So u_x and u need to satisfy $u_x + \sqrt{\lambda}(u^2 - a^2) = 0$, which implies

$$\int \sqrt{\lambda} dx = \int \frac{du}{(a^2 - u^2)} = \frac{1}{a}\text{arctanh}\left(\frac{u}{a}\right).$$

This implies $u = a \tanh\left[a\sqrt{\lambda}(x - x_0)\right]$ where x_0 is a constant of integration, which indeed matches the solution with lowest energy found above.

17. (a) Explain why the Bogomol'nyi argument given in the lectures fails to provide a useful bound on the energy of a two-kink solution of the sine-Gordon equation (a two-kink solution is one with topological charge $n - m$ equal to 2). What is the most that can be said about the energy of a k-kink?

(b) For a sine-Gordon field u, generalise the Bogomol'nyi argument to show that

$$\int_A^B dx \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + (1 - \cos u)\right] \geq \pm 4 \left[\cos \frac{u}{2}\right]_A^B .$$

(c) * Use this result and the intermediate value theorem (look it up if necessary!) to show that if the field u has the boundary conditions of a k-kink, then its energy is at least k times that of a single kink. Can this bound be saturated?

Solution

(a) The end result of the Bogomolnyi argument for sine-Gordon was the following inequality for the total energy $E = T + V$ of a field configuration u:

$$E \geq \pm 4 \left[\cos\left(\tfrac{1}{2}u\right)\right]_{-\infty}^{\infty} , \qquad (*)$$

with equality if and only if $u_t = 0$ and

$$\int_{-\infty}^{\infty} \tfrac{1}{2}\left(u_x \pm 2\sin\left(\tfrac{1}{2}u\right)\right)^2 dx = 0 .$$

If u has the boundary conditions of a two-kink, then $u|_{x=+\infty} = u|_{x=-\infty} + 4\pi$, and since $\left[\cos\left(\frac{1}{2}u\right)\right]_{-\infty}^{\infty} = 0$, no matter what sign is taken in $(*)$, we can't deduce any more from it than $E \geq 0$, which we knew already. On the other hand, if u was a 3-kink, then $u|_{x=+\infty} = u|_{x=-\infty} + 6\pi$ and we would have $\left[\cos\left(\frac{1}{2}u\right)\right]_{-\infty}^{\infty} = \pm 2$ (just as for the one-kink) and so for one or other choice of sign we would have $E \geq 8$, the energy of a one-kink. In general, if n is odd, we can deduce $E \geq 8$, while if n is even the situation is the same as for the 2-kink – we can't say anything.

(b) To prove the new inequality, just repeat the arguments in lectures, but with all integrals from A to B instead of from $-\infty$ to $+\infty$:

$$\begin{aligned}\int_A^B \left(\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + (1-\cos u)\right) dx \\ &\geq \int_A^B \frac{1}{2}(u_x)^2 + (1-\cos u)\, dx \\ &= \int_A^B \frac{1}{2}(u_x)^2 + 2\sin^2(\frac{1}{2}u)\, dx \\ &= \int_A^B \frac{1}{2}\left(u_x \pm 2\sin\left(\frac{1}{2}u\right)\right)^2 \mp 2\sin\left(\frac{1}{2}u\right) u_x\, dx \\ &= \int_A^B \frac{1}{2}\left(u_x \pm 2\sin\left(\frac{1}{2}u\right)\right)^2 dx \pm 4\left[\cos\left(\frac{1}{2}u\right)\right]_A^B\end{aligned}$$

Since the thing being integrated in the last line is non-negative (it's a perfect square) the required inequality now follows.

(c) Suppose we have n-kink boundary conditions, with $u(x,t)|_{x=-\infty} = 0$ and $u(x,t)|_{x=+\infty} = 2n\pi$. By the intermediate value theorem, the value of $u(x,t)$ must pass through all values between 0 and $2n\pi$ as x varies from $-\infty$ to $+\infty$. In particular, u must take the values 2π, 4π, … $2(n-1)\pi$, say at the points x_1, x_2, … x_{n-1}, so that $u(x_k) = 2k\pi$. Setting $x_0 = -\infty$ and $x_n = +\infty$, the integral giving the total energy of a configuration with n-kink boundary conditions can be split into n pieces:

$$E = \int_{-\infty}^{+\infty} \mathcal{E}\, dx = \int_{x_0}^{x_1} \mathcal{E}\, dx + \int_{x_1}^{x_2} \mathcal{E}\, dx + \cdots \int_{x_{n-1}}^{x_n} \mathcal{E}\, dx$$

where $\mathcal{E} = \frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + (1-\cos u)$ is the energy density. Now each of the n integrals on the RHS has 1-kink boundary conditions, and $4[\cos(\frac{1}{2}u)]_{x_k}^{x_{k+1}} = \pm 8$ for each k. Using the result from part (b), for one or other sign, each of these sub-integrals must therefore be larger than or equal to 8, and so the sum of n of them must be larger than or equal to $8n$, which is n times the energy of a single kink.
To saturate the bound, we'd need $u_t = 0$ (which is OK) and also $u_x \pm 2\sin(\frac{1}{2}u) = 0$ on each subinterval $[x_k, x_{k+1}]$. This is the Bogomolnyi equation, which was solved in lectures – where it was found that the only places where the solution attained a value which was an integer multiple of 2π were $x = -\infty$ and $x = +\infty$. However on the subinterval $[x_k, x_{k+1}]$ we must have $u(x_k) = 2k\pi$ and $u(x_{k+1}) = 2(k+1)\pi$, so u must depart somewhere from the solution of the Bogomolnyi equation, and hence the bound cannot be saturated. In fact the bounding value of E is approached in the limit where the 1-kinks making up the solution become infinitely far apart – which is a sign that single sine-Gordon kinks repel each other.

18. A system on the finite interval $-\pi/2 \leq x \leq \pi/2$ is defined by the following expressions for the kinetic energy T and the potential energy V:

$$T = \int_{-\pi/2}^{\pi/2} dx\, \frac{1}{2}u_t^2\,, \qquad V = \int_{-\pi/2}^{\pi/2} dx\, \frac{1}{2}\left(u_x^2 + 1 - u^2\right)\ .$$

The function $u(x,t)$ satisfies the boundary condition $|u(\pm\pi/2,t)| = 1$ and is required to satisfy $|u(x,t)| \leq 1$ everywhere. Show that with "kink" boundary conditions, the total energy E is bounded below by a positive constant, and find a solution for which the bound is saturated.

Solution This is left for you to try!

19. Check explicitly that the energy

$$E = \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \mathbb{V}(u)\right]$$

and the momentum

$$P = -\int_{-\infty}^{+\infty} dx \, u_t u_x$$

of a relativistic field $u(x,t)$ in 1 space and 1 time dimensions are conserved when the equation of motion

$$u_{tt} - u_{xx} = -\mathbb{V}'(u)$$

and the boundary conditions

$$u_t, \; u_x, \; \mathbb{V}(u), \; \mathbb{V}'(u) \underset{x\to\pm\infty}{\longrightarrow} 0 \qquad \forall t$$

are satisfied.

Solution Defining $\mathcal{E} = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \mathbb{V}(u)$ we have

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} &= u_t u_{tt} + u_x u_{xt} + \mathbb{V}'(u) \cdot u_t \\
&= u_t(u_{tt} + \mathbb{V}'(u)) + u_x u_{xt} \\
&\underset{\text{EoM}}{=} u_t u_{xx} + u_x u_{xt} = \frac{\partial}{\partial x}(u_t u_x) \equiv \frac{\partial}{\partial x}(-j) \, ,
\end{aligned}$$

where $j \equiv -u_t u_x$. Since with the given boundary conditions j has the same (zero) limit at $\pm\infty$, this shows that E is conserved.

Next set $\mathcal{P} = -u_t u_x$. Then

$$\begin{aligned}
\frac{\partial \mathcal{P}}{\partial t} &= -u_{tt} u_x - u_t u_{xt} \\
&\underset{\text{EoM}}{=} (\mathbb{V}'(u) - u_{xx}) u_x - u_t u_{xt} \\
&= \frac{\partial}{\partial x}\left(\mathbb{V}(u) - \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2\right) .
\end{aligned}$$

The quantity in brackets on the last line vanishes at $\pm\infty$, and so, by the standard argument, P is also conserved.

20. (a) Compute the conserved topological charge, energy and momentum of a sine-Gordon kink moving with velocity v, and check that the results do not depend on time. [**Hint**: The integral sheet might be useful. For the scalar potential term in the energy, write $1 - \cos(u) = 2\sin^2(u/2)$, plug in the kink solution and manipulate the result to get something involving $\cosh^{-2}$.]
Confirm that for $|v| \ll 1$ the energy and the momentum take the forms

$$E = M + \frac{1}{2}Mv^2 + \mathcal{O}(v^4)\,, \qquad P = Mv + \mathcal{O}(v^3)$$

where the 'mass' M is the energy of the static kink, which appears in the Bogomol'nyi bound.

(b) * If you are fearless and have time on your hands, try also to compute the conserved spin 3 charge

$$Q_3 = \int_{-\infty}^{+\infty} dx\ \left[u_{++}^2 - \frac{1}{4}u_+^4 + u_+^2 \cos u\right]$$

for the sine-Gordon kink. The integrals are not at all straightforward, but can be evaluated using appropriate changes of variables. (Did I write fearless?)

Solution The sine-Gordon kink moving with velocity v is

$$u(x,t) = 4\arctan\left(e^{\gamma(x-x_0-vt)}\right)\,.$$

Its topological charge is

$$Q_0 = \frac{1}{2\pi}\,[u]_{-\infty}^{+\infty} = \frac{2\pi - 0}{2\pi} = 1\,.$$

(This is normalised so that kinks/antikinks have topological charge ± 1.)

The energy is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + (1-\cos u)\,dx = \int_{-\infty}^{\infty} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + 2\sin^2 u/2\,dx \\ &= \int_{-\infty}^{\infty} 2(1{+}v^2)\gamma^2\,\mathrm{sech}^2(\gamma(x-x_0-vt)) + 2\sin^2(2\arctan\left(e^{\gamma(x-x_0-vt)}\right))\,dx \\ &= \frac{1}{\gamma}\int_{-\infty}^{\infty} 2\frac{1{+}v^2}{1{-}v^2}\,\mathrm{sech}^2(y) + 2\sin^2(2\arctan\left(e^y\right))\,dx\,. \end{aligned}$$

Let $\theta \equiv \arctan(e^y)$. Then the last term, $2\sin^2(2\theta)$, is equal to

$$8\sin^2\theta\cos^2\theta = 8\tan^2\theta\cos^4\theta = \frac{8\tan^2\theta}{(\tan^2\theta+1)^2} = \frac{8}{(\tan\theta+\tan^{-1}\theta)^2} = 2\,\mathrm{sech}^2\,y\,.$$

Hence

$$E = \frac{1}{\gamma}\int_{-\infty}^{\infty} 2\left(\frac{1{+}v^2}{1{-}v^2}+1\right)\mathrm{sech}^2(y)\,dx = 4\gamma\int_{-\infty}^{\infty}\mathrm{sech}^2(y)\,dx = 8\gamma = \frac{8}{\sqrt{1-v^2}}\,.$$

The momentum is an easier calculation:

$$P = -\int_{-\infty}^{\infty} u_t u_x \, dx = v \int_{-\infty}^{\infty} u_x^2 \, dx$$
$$= 4v\gamma^2 \int_{-\infty}^{\infty} \text{sech}^2(\gamma(x - x_0 - vt)) \, dx$$
$$= 4\gamma v \int_{-\infty}^{\infty} \text{sech}^2(y) \, dx = 8\gamma v \, .$$

All three (topological charge, energy and momentum) are clearly independent of time, as indeed had to be the case. Finally, since the energy M of a static kink is equal to 8, the Taylor expansion

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + \ldots$$

is enough to derive the claimed formule for E and P for $|v| \ll 1$.

21. Find three conserved charges for the mKdV equation of problem 13 (a), which involve u, u^2 and u^4 respectively. The boundary conditions on $u(x,t)$ are u, u_x and $u_{xx} \to 0$ as $|x| \to \infty$. Evaluate these quantities for the travelling-wave solution found in that problem. The definite integrals on the integrals sheet might help.

Solution

- The mKdV equation can be written as a continuity equation

$$u_t + (2u^3 + u_{xx})_x = 0 \, ,$$

where we identify the charge density $\rho_1 = u$ and the current density $j_1 = 2u^3 + u_{xx}$. The BC's imply that $j \to 0$ as $x \to \pm\infty$, so the charge

$$Q_1 = \int_{-\infty}^{\infty} dx \; u$$

is conserved.

- Now we try $\rho_2 = u^2$. To see if it satisfies a continuity equation $(\rho_2)_t + (j_2)_x = 0$ with a suitable current j_2, let's calculate

$$(u^2)_t = 2uu_t \underset{\text{mKdV}}{=} -12u^3 u_x - 2uu_{xxx} = (-3u^4 - 2uu_{xx})_x + 2u_x u_{xx}$$
$$= -(3u^4 + 2uu_{xx} - u_x^2)_x \, .$$

We identify $j_2 = 3u^4 + 2uu_{xx} - u_x^2$, which has the same limit (equal to zero) as $x \to \pm\infty$. [**NOTE:** It is fine to drop x-derivatives of functions which have the same limits at $\pm\infty$ as I did in lectures. Here I keep track of the current even though we only care that it has the same limits at $\pm\infty$.]
Therefore the charge

$$Q_2 = \int_{-\infty}^{\infty} dx \; u^2$$

is conserved.

- Now we try $\rho_4 = u^4$, and calculate

$$\begin{aligned}(u^4)_t &= 4u^3 u_t \underset{\text{mKdV}}{=} -24u^5 u_x - 4u^3 u_{xxx} = (-4u^6 - 4u^3 u_{xx})_x + 12u^2 u_x u_{xx} \\ &\underset{\text{mKdV}}{=} -(4u^6 + 4u^3 u_{xx})_x - 2(u_t + u_{xxx})u_{xx} \\ &= -(4u^6 + 4u^3 u_{xx} + 2u_t u_x + u_{xx}^2)_x + 2u_{tx} u_x \\ &= -(4u^6 + 4u^3 u_{xx} + 2u_t u_x + u_{xx}^2)_x + (u_x^2)_t \ .\end{aligned}$$

The right-hand side is not an x-derivative, but we can bring the time derivative to the left-hand side to write this as a continuity equation

$$(\underbrace{u^4 - u_x^2}_{=\rho_4})_t + (\underbrace{4u^6 + 4u^3 u_{xx} + 2u_t u_x + u_{xx}^2}_{=j_4})_x = 0.$$

The BC's imply that $j_4 \to 0$ as $x \to \pm\infty$, so the charge

$$Q_4 = \int_{-\infty}^{\infty} dx \ \left(u^4 - u_x^2\right)$$

is conserved.

To evaluate these charges for the travelling wave solutions, note that the integration variable can be shifted $x \mapsto x + x_0 + vt$ to get rid of the integration constant x_0 and the time t in the charges. (Since the charges are conserved, they shouldn't depend on time anyway – but here we see this fact directly.) So we can simply take

$$u(x) = \pm v^{1/2} \ \mathrm{sech}\big(v^{1/2} x\big)$$

in the following. We will also use the integrals

$$I_1 = \int_{-\infty}^{\infty} dy \ \mathrm{sech}(y) = \pi \ , \quad I_2 = \int_{-\infty}^{\infty} dy \ \mathrm{sech}^2(y) = 2 \ , \quad I_4 = \int_{-\infty}^{\infty} dy \ \mathrm{sech}^4(y) = \frac{4}{3} \ ,$$

which can be extracted from the table of integrals.

We easily calculate

$$\begin{aligned}Q_1 &= \pm v^{1/2} \int_{-\infty}^{\infty} dx \ \mathrm{sech}\big(v^{1/2} x\big) = \pm \int_{-\infty}^{\infty} dy \ \mathrm{sech}(y) = \pm\pi \ , \\ Q_2 &= v \int_{-\infty}^{\infty} dx \ \mathrm{sech}^2(v^{1/2} x) = v^{1/2} \int_{-\infty}^{\infty} dy \ \mathrm{sech}^2(y) = 2v^{1/2} \ ,\end{aligned}$$

where the second equality in both lines follows from setting $y = v^{1/2} x$. Note that the measure changes: $dx = v^{-1/2} dy$. This is a common source of errors.

For Q_4, we first calculate

$$u_x = \mp v \sinh\big(v^{1/2} x\big) \cdot \mathrm{sech}^2(v^{1/2} x) \ ,$$

then

$$\begin{aligned} Q_4 &= v^2 \int_{-\infty}^{\infty} dx\ \mathrm{sech}^4(v^{1/2}x)\cdot(1-\sinh^2(v^{1/2}x)) \\ &= v^{3/2}\int_{-\infty}^{\infty} dy\ \mathrm{sech}^4(y)\cdot(2-\cosh^2(y)) \\ &= v^{3/2}\int_{-\infty}^{\infty} dy\ (2\,\mathrm{sech}^4(y)-\mathrm{sech}^2(y)) \\ &= v^{3/2}(2I_4-I_2) = v^{3/2}\left(\frac{8}{3}-2\right) = \frac{2}{3}v^{3/2}\,. \end{aligned}$$

22. Show that u is a conserved density for Burgers' equation from problem 13 (e). Why is this result of no use in analysing the travelling wave solution of that problem?

    Solution The equation itself can be rewritten to show that u is a conserved density:

    $$u_t + uu_x - u_{xx} = 0 \quad \Leftrightarrow \quad u_t + (\tfrac{1}{2}u^2 - u_x)_x = 0\,.$$

    However the 'j' for this would-be conservation law has different limits at $\pm\infty$ for the boundary conditions given in the problem, so we can't deduce that the correspondng charge is conserved (an in fact it is formally infinite for the travelling wave solution).

23. Consider the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$.

    (a) Show that $\rho_1 \equiv u$, $\rho_2 \equiv u^2$ and $\rho_* \equiv xu - 3tu^2$ are all conserved densities, so that

    $$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_* = \int_{-\infty}^{+\infty} dx\ (xu - 3tu^2)$$

    are all conserved charges.

    (b) Evaluate the conserved charges Q_1, Q_2 and Q_* for the one-soliton solution centred at x_0 and moving with velocity $v = 4\mu^2$:

    $$u_{\mu,x_0}(x,t) = 2\mu^2\,\mathrm{sech}^2\left[\mu(x - x_0 - 4\mu^2 t)\right]\ .$$

    (c) According to the KdV equation, the initial condition $u(x,0) = 6\ \mathrm{sech}^2(x)$ is known to evolve into the sum of two well-separated solitons with different velocities $v_1 = 4\mu_1^2$ and $v_2 = 4\mu_2^2$ at late times. Use the conservation of Q_1 and Q_2 to determine v_1 and v_2.

    (d) A two-soliton solution separates as $t \to -\infty$ into two one-solitons u_{μ_1,x_1} and u_{μ_2,x_2}. As $t \to +\infty$, two one-solitons are again found, with μ_1 and μ_2 unchanged but with x_1, x_2 replaced by y_1, y_2. Use the conservation of Q_* to find a formula relating the *phase shifts* $y_1 - x_1$ and $y_2 - x_2$ of the two solitons.

Solution

(a) The calculations showing that u and u^2 are conserved densities were done in lectures. For $Q_* = \int_{-\infty}^{+\infty} dx\, (xu - 3tu^2)\, dx$, consider

$$\begin{aligned}\left(xu - 3tu^2\right)_t &= xu_t - 3u^2 - 6tuu_t \\ &= -x(6uu_x + u_{xxx}) - 3u^2 + 6tu(6uu_x + u_{xxx})\,,\end{aligned}$$

using the KdV equation. Collecting some total x derivatives, we have:

$$\begin{aligned}6xuu_x &= (3xu^2)_x - 3u^2 \\ xu_{xx} &= (xu_x)_x - u_x \\ 36tu^2u_x &= (12tu^3)_x \\ 6tuu_{xxx} &= (6tuu_{xx})_x - (3tu_x^2)_x,\end{aligned}$$

and so

$$\left(xu - 3tu^2\right)_t = -\Bigg(\underbrace{3xu^2 + xu_{xx} - u_x - 12tu^3 - 6tuu_{xx} + 3tu_x^2}_{X(t,x,u,u_x,u_{xx})}\Bigg)_x\,.$$

(b) We have

$$\begin{aligned}Q_1[u_{\mu,x_0}] &= \int_{-\infty}^{+\infty} dx\, u_{\mu,x_0}(x,t)\, dx \\ &= 2\mu^2 \int_{-\infty}^{+\infty} dx\ \operatorname{sech}^2(\mu(x - x_0 - 4\mu^2 t))\, dx \\ &= 2\mu^2 \int_{-\infty}^{+\infty} dx\ \operatorname{sech}^2(\mu x)\, dx \\ &= 2\mu \int_{-\infty}^{+\infty} dx\ \operatorname{sech}^2(x)\, dx \ = \ 4\mu\end{aligned}$$

using one of the integrals from earlier for the final equality. Similarly,

$$\begin{aligned}Q_2[u_{\mu,x_0}] &= \int_{-\infty}^{+\infty} dx\, u_{\mu,x_0}(x,t)^2\, dx \\ &= 4\mu^4 \int_{-\infty}^{+\infty} dx\ \operatorname{sech}^4(\mu(x - x_0 - 4\mu^2 t))\, dx \\ &= 4\mu^4 \int_{-\infty}^{+\infty} dx\ \operatorname{sech}^4(\mu x)\, dx \\ &= 4\mu^3 \int_{-\infty}^{+\infty} dx\ \operatorname{sech}^4(x)\, dx \ = \ \frac{16}{3}\mu^3.\end{aligned}$$

The third charge is related to Q_2 via

$$\begin{aligned}Q_*[u_{\mu,x_0}] &= \int_{-\infty}^{+\infty} dx\, xu_{\mu,x_0}(x,t)\, dx - 3tQ_2[u_{\mu,x_0}] \\ &= \int_{-\infty}^{+\infty} dx\, 2\mu^2 x\ \operatorname{sech}^2(\mu(x - x_0 - 4\mu^2 t))\, dx - 3tQ_2[u_{\mu,x_0}]\,.\end{aligned}$$

Substituting $y = x - x_0 - 4\mu^2 t$ in the integral on the RHS,

$$\int_{-\infty}^{+\infty} dx\, 2\mu^2 x\, \mathrm{sech}^2(\mu(x-x_0-4\mu^2 t))\, dx$$
$$= \int_{-\infty}^{+\infty} dx\, 2\mu^2 y\, \mathrm{sech}^2(\mu y)\, dy + (x_0+4\mu^2 t) \int_{-\infty}^{+\infty} dx\, \mu^2\, \mathrm{sech}^2(\mu y)\, dy.$$

The first integral vanishes, since it is the (convergent) integral of an odd function over an interval symmetric with respect to 0. (Please check you understand why the integral converges!) The second one is just $(x_0 + 4\mu^2 t) Q_1[u_{\mu,x_0}]$, and so:

$$\begin{aligned} Q_*[u_{\mu,x_0}] &= (x_0 + 4\mu^2 t) Q_1[u_{\mu,x_0}] - 3t Q_2[u_{\mu,x_0}] \\ &= (x_0 + 4\mu^2 t)(4\mu) - 3t(\frac{16}{3}\mu^3) \\ &= 4\mu x_0. \end{aligned}$$

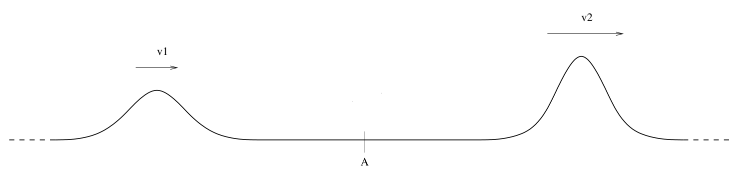
(c) Doing the same calculation as in part (b) for $u(x, t{=}0) = 6\, \mathrm{sech}^2(x)$, you should find $Q_1 = 12$ and $Q_2 = 48$. Since Q_1 and Q_2 are conserved, they must also have these values later, when u looks like a pair of well-separated solitons with parameters μ_1 and μ_2. But for the well-separated solitons the values of Q_1 and Q_2 must just be the *sum* of the one-soliton values for μ_1 and μ_2. Indeed, a charge Q associated to a two-soliton solution is given by the integral of the two-soliton density $T^{(2)}$, which can be be split into two contributions:

$$Q = \int_{-\infty}^{A(t)} T^{(2)}\, dx + \int_{A(t)}^{+\infty} T^{(2)}\, dx$$

If the point $A(t)$ sits between the two solitons (moving faster than the first soliton, but slower than the second), then as $t \to \infty$, the field in the two integration regions simplifies to:

$$u(x,t) = \begin{cases} u_1(x,t) & : \quad x \le A(t) \\ u_2(x,t) & : \quad x \ge A(t) \end{cases}$$

where u_1 and u_2 are the one-soliton solutions.



The charges become additive as $t \to \infty$.

The charge Q is then simply the sum of two one-soliton contributions $T^{(1)}$:

$$Q = \int_{-\infty}^{A(t)} T_1^{(1)}\, dx + \int_{A(t)}^{+\infty} T_2^{(1)}\, dx.$$

But $u_1 \approx 0$ (resp. $u_2 \approx 0$) for $x \ge A$ (resp. $x \le A$), and therefore each each integral in the previous equation can be extended to the whole line:

$$Q = \int_{-\infty}^{+\infty} dx\, T_1^{(1)}\, dx + \int_{-\infty}^{+\infty} dx\, T_2^{(1)}\, dx,$$

which shows that Q is indeed the sum of the two asymptotic one-soliton charges. [Technical remark: the fact that Q is exactly conserved means that we can wait arbitrarily long before evaluating the one-soliton charges, meaning that the errors caused by the approximations used above can be made arbitrarily small – in other words, the addititive property holds *exactly*.] In this case, the conservation of Q_1 implies that $Q_1\big|_{t\to+\infty} = Q_1\big|_{t=0}$, or in other words

$$4\mu_1 + 4\mu_2 = 12$$

while the conservation of Q_2 similarly implies

$$\frac{16}{3}\mu_1^3 + \frac{16}{3}\mu_2^3 = 48\,.$$

Hence $\mu_1 + \mu_2 = 3$ and $\mu_1^3 + \mu_2^3 = 9$. Rewriting the second equation, $(\mu_1 + \mu_2)(\mu_1^2 - \mu_1\mu_2 + \mu_2^2) = 9$ or, using the first equation, $\mu_1^2 - \mu_1\mu_2 + \mu_2^2 = 3$. The first equation also implies $\mu_2 = 3 - \mu_1$ and solving you should find μ_1 equals either 1 or 2, and hence $\mu_2 = 2$ or 1. Hence the final velocities are 4 and 16. (Note that this is consistent with the statement of exercise 25.)

(d) Using the additivity of Q_3 at $t \to \pm\infty$,

$$Q_3[u_{\mu_1,x_1}] + [u_{\mu_2,x_2}] = Q_3[u_{\mu 1,y_1}] + Q_3[u_{\mu_2,y_2}],$$

and we obtain the following constraint:

$$\mu_1 x_1 + \mu_2 x_2 = \mu_1 y_1 + \mu_2 y_2.$$

This turns out to be consistent with the exact formula for the phase shift induced by the interaction of two KdV solitons, but since we didn't cover multi-soliton solutions of KdV in lectures, you'll have to look in the book by Drazin and Johnson (for example) for that.

24. (a) Show that if $u(x,t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where λ is a constant and $v(x,t)$ some other function, then v satisfies

$$\big(2v + \frac{\partial}{\partial x}\big)\big(v_t + 6\lambda v_x - 6v^2 v_x + v_{xxx}\big) = 0\,.$$

(b) Compute the Gardner transform expansion

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t)\varepsilon^n$$

up to order ε^4. Use the results to find the conserved charges $\widetilde{Q}_3$ and $\widetilde{Q}_4$, where

$$\widetilde{Q}_n = \int_{-\infty}^{+\infty} dx\; w_n\,.$$

Show that $\widetilde{Q}_3$ is the integral of a total x-derivative (and hence is zero), while $\widetilde{Q}_4 = \alpha\, Q_3$, where

$$Q_3 = \int_{-\infty}^{+\infty} dx\, \left(u^3 - \frac{1}{2}u_x^2\right)$$

is the third KdV conserved charge (the 'energy') and α a constant that you should determine. * If you're feeling energetic, try to compute $\widetilde{Q}_5$ and $\widetilde{Q}_6$ as well.

Solution

(a) Differentiating $u = \lambda - v^2 - v_x$ yields:

$$\begin{aligned} u_t &= -2vv_t - v_{tx} \\ u_x &= -2vv_x - v_{xx} \\ u_{xx} &= -2v_x^2 - 2vv_{xx} - v_{xxx} \\ u_{xxx} &= -6v_xv_{xx} - 2vv_{xxx} - v_{xxxx}\,. \end{aligned}$$

Substituting into the KdV equation, and noting that $(v^2v_x)_x = v^2v_{xx} + 2vv_x^2$, we find

$$-2v\left[v_t + 6\lambda v_x + 6v^2v_x + v_{xxx}\right] - \frac{\partial}{\partial x}\left[v_t + 6\lambda v_x + 6v^2v_x + v_{xxx}\right] = 0,$$

and thus

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t + 6\lambda v_x + 6v^2v_x + v_{xxx}\right) = 0\,.$$

(b) The Gardner transform defines w implicitly in terms of u as follows:

$$u = -w - \varepsilon w_x - \varepsilon^2 w^2$$

To solve for w, we write it as a (formal) power series in ε, $w = \sum_{n=0}^{\infty} w_n\varepsilon^n$, and equate both sides order by order in ε. At order ε^0 this gives $w_0 = -u$; after that, since the LHS is independent of ε, the terms on the RHS depending on ε^n must sum to zero for any $n > 0$. Let us do this energetically up to order 6. We have (notice where truncations are made):

$$\varepsilon\, w_x = \sum_{n=0}^{5} w_{nx}\, \varepsilon^{n+1} + \mathcal{O}(\varepsilon^7),$$

and

$$\begin{aligned} \varepsilon^2\, w^2 &= \varepsilon^2\, (\sum_{n=0}^{4} w_n\varepsilon^n)^2 + \mathcal{O}(\varepsilon^7) \\ &= \varepsilon^2\, w_0^2 + 2w_0w_1\varepsilon^3 + (w_1^2 + 2w_0w_2)\varepsilon^4 + 2(w_0w_3 + w_1w_2)\varepsilon^5 \\ &\quad + (w_2^2 + 2w_0w_4 + 2w_1w_3)\varepsilon^6 + \mathcal{O}(\varepsilon^7)\,. \end{aligned}$$

It follows that:

$$\begin{aligned}
w_0 &= -u \\
w_1 &= -w_{0x} \\
w_2 &= -w_{1x} - w_0^2 \\
w_3 &= -w_{2x} - 2w_0 w_1 \\
w_4 &= -w_{3x} - w_1^2 - 2w_0 w_2 \\
w_5 &= -w_{4x} - 2w_0 w_3 - 2w_1 w_2 \\
w_6 &= -w_{5x} - w_2^2 - 2w_0 w_4 - 2w_1 w_3 .
\end{aligned}$$

Solving recursively in terms of u:

$$\begin{aligned}
w_0 &= -u \\
w_1 &= u_x \\
w_2 &= -u_{xx} - u^2 \\
w_3 &= u_{xxx} + 4uu_x \\
w_4 &= -u_{xxxx} - 5u_x^2 - 6uu_{xx} - 2u^3 \\
w_5 &= u_{xxxxx} + 18u_x u_{xx} + 8uu_{xxx} + 16u^2 u_x \\
w_6 &= -u_{xxxxxx} - 19u_{xx}^2 - 28u_x u_{xxx} - 10uu_{xxxx} \\
&- 50uu_x^2 - 30u^2 u_{xx} - 5u^4 .
\end{aligned}$$

Now, w_3 is indeed a total x derivative:

$$w_3 = -(u_{xx} + 2u^2)_x \,,$$

and therefore $\widetilde{Q}_3 = 0$. On the other hand,

$$w_4 = -2(u^3 - \frac{1}{2}u_x^2) - (u_{xx} + 6uu_x)_x \,,$$

so that $\widetilde{Q}_4 = -2Q_3$.
Scrutiny of w_5 shows that it too is a total x derivative:

$$w_5 = (u_{xxxx} + 9u_x^2 + 8uu_{xx} - 4u_x^2 + \frac{16}{3}u^3)_x$$

and hence $\widetilde{Q}_5 = 0$. For w_6, we have

$$\begin{aligned}
w_6 = -5(u^4 - 2uu_x^2 + \frac{1}{2}u_{xx}^2) \\
- (u_{xxxxx} + 10uu_{xxx} + 18u_x u_{xx} + 25u^2 u_x + 5u^2 u_x)_x \,,
\end{aligned}$$

and we recognize the fourth KdV conserved charge:

$$\widetilde{Q}_6 = -5\int_{-\infty}^{+\infty} \left(u^4 - 2uu_x^2 + \frac{1}{2}u_{xx}^2\right) dx = -5Q_4 \,.$$

25. This question is also about the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

(a) Evaluate the first three KdV conserved charges

$$Q_1 = \int_{-\infty}^{+\infty} dx\, u\,, \qquad Q_2 = \int_{-\infty}^{+\infty} dx\, u^2\,, \qquad Q_3 = \int_{-\infty}^{+\infty} dx\, \left(u^3 - \frac{1}{2}u_x^2\right)$$

for the initial state $u(x,0) = A\ \mathrm{sech}^2(Bx)$, where A and B are constants.

(b) The initial state

$$u(x,0) = N(N+1)\ \mathrm{sech}^2(x)\,,$$

where N is an integer, is known to evolve at late times into N well-separated solitons, with velocities $4k^2$, $k = 1 \ldots N$. So for $t \to +\infty$, this solution approaches the sum of N single well-separated solitons

$$u(x,t) \approx \sum_{k=1}^{N} 2\mu_k^2\ \mathrm{sech}^2\left[\mu_k(x - x_k - 4\mu_k^2 t)\right]\,,$$

where $\mu_1, \ldots, \mu_N$ are N different constants. Since Q_1, Q_2 and Q_3 are conserved, their values at $t = 0$ and $t \to +\infty$ must be equal. Use this fact to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers.

(c) * Use Q_4 and Q_5 and the method just described to find the sum of the first N seventh and ninth powers, $\sum_{k=1}^{N} k^7$ and $\sum_{k=1}^{N} k^9$.

Solution

(a) We can use the definite integrals

$$I_n := \int_{-\infty}^{\infty} dy\ \mathrm{sech}^{2n}(y) = \frac{2^{2n-1}((n-1)!)^2}{(2n-1)!}$$
$$\implies I_1 = 2, \quad I_2 = \frac{4}{3}\,, \quad I_3 = \frac{16}{15}\,, \quad I_4 = \frac{32}{35}\,, \quad I_5 = \frac{256}{315}\,, \quad \ldots$$

which are tabulated on the integrals sheet, as well as the derivative formula

$$\frac{d}{dy}\mathrm{sech}\ y = -\mathrm{sech}\ y \cdot \tanh y = -\mathrm{sech}^2\, y \cdot \sinh y\,,$$

which implies

$$\frac{d}{dy}\mathrm{sech}^2\, y = -2\ \mathrm{sech}^3\, y \cdot \sinh\ y$$
$$\left(\frac{d}{dy}\mathrm{sech}^2\, y\right)^2 = 4\ \mathrm{sech}^6\, y \cdot \sinh^2 y = 4\ \mathrm{sech}^6\, y \cdot (\cosh^2 y - 1) = 4\,(\mathrm{sech}^4\, y - \mathrm{sech}^6\, y)\,.$$

We can then evaluate Q_1, Q_2, Q_3 at $t = 0$ by changing integration variable $y = Bx$:

$$Q_1 = \frac{A}{B}I_1 = \frac{2A}{B}$$
$$Q_2 = \frac{A^2}{B}I_2 = \frac{4A^2}{3B}$$
$$Q_3 = \frac{A^3}{B}I_3 - 2A^2B(I_2 - I_3) = \frac{8A^2(2A - B^2)}{15B}\,.$$

(b) To calculate the conserved charges Q_1, Q_2, Q_3 at the initial time $t = 0$, we can just set $A = N(N+1)$ and $B = 1$ in the results of part 1:

$$\begin{aligned} Q_1 &= 2N(N+1) \\ Q_2 &= \frac{4}{3}N^2(N+1)^2 \\ Q_3 &= \frac{8}{15}N^2(N+1)^2(2N^2+2N-1)\,. \end{aligned}$$

To calculate the conserved charges Q_1, Q_2, Q_3 at late times $t \to +\infty$, we use the fact that the N solitons are well-separated, that is, they are separated by much larger distances than the widths of the solitons. So

$$u \approx \sum_{k=1}^{N} u_{k,x_k}\,, \qquad u^2 \approx \sum_{k=1}^{N} u_{k,x_k}^2\,, \qquad u^3 \approx \sum_{k=1}^{N} u_{k,x_k}^3\,, \qquad u_x^2 \approx \sum_{k=1}^{N} (\partial_x u_{k,x_k})^2$$

where

$$u_{k,x_k}(x,t) = 2k^2\,\mathrm{sech}^2\left[k(x - x_k - 4k^2 t)\right]\,.$$

The cross terms are negligible because the soliton solutions u_{k,x_k} tend to zero exponentially fast away from their centres, and the solitons are assumed to be well-separated. So the contributions of the N well-separated solitons simply add up. Comparing with part 1, the k-th soliton u_{k,x_k} has $A_k = 2k^2$ and $B_k = k$ (the shift of x by $x_k + 4k^2 t$ is inconsequential for the calculation of the charges as it can be absorbed by a shift of the integration variable.) So we find the charges

$$\begin{aligned} Q_1 &= 2\sum_{k=1}^{N} \frac{A_k}{B_k} = 4\sum_{k=1}^{N} k \\ Q_2 &= \frac{4}{3}\sum_{k=1}^{N} \frac{A_k^2}{B_k} = \frac{16}{3}\sum_{k=1}^{N} k^3 \\ Q_3 &= \frac{8}{15}\sum_{k=1}^{N} \frac{A_k^2(2A_k - B_k^2)}{B_k} = \frac{32}{5}\sum_{k=1}^{N} k^5\,. \end{aligned}$$

Equating the $t = 0$ expressions with the $t \to +\infty$ expressions for the conserved charges we find

$$\sum_{k=1}^{N} k = \frac{1}{2}N(N+1)\,, \quad \sum_{k=1}^{N} k^3 = \frac{1}{4}N^2(N+1)^2\,, \quad \sum_{k=1}^{N} k^5 = \frac{1}{12}N^2(N+1)^2(2N^2+2N-1)\,.$$

(c) We need to use the next two conserved charges

$$\begin{aligned} Q_4 &= \int_{-\infty}^{\infty} dx\, \left(u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2\right) \\ Q_5 &= \int_{-\infty}^{\infty} dx\, \left(u^5 - 5u^2u_x^2 + uu_{xx}^2 - \frac{1}{14}u_{xxx}^2\right)\,, \end{aligned}$$

which again we would like to evaluate for the field configuration $u = A\ \text{sech}^2(Bx)$.

If the integrands can be expressed in terms of even powers of $z \equiv \text{sech}(y)$, where $y = Bx$, the integrals I_n can be used. There are a few ways to do that; here's one. Using the chain rule,

$$\frac{d}{dy} = \frac{dz}{dy}\frac{d}{dz} = -z\sqrt{1-z^2}\frac{d}{dz}$$
$$\frac{d^2}{dy^2} = \left(-z\sqrt{1-z^2}\frac{d}{dz}\right)^2 = z^2(1-z^2)\frac{d^2}{dz^2} + z(1-2z^2)\frac{d}{dz}$$
$$\frac{d^3}{dy^3} = \left(-z\sqrt{1-z^2}\frac{d}{dz}\right)^3 = z\sqrt{1-z^2}\left[z^2(1-z^2)\frac{d^3}{dz^3} + 3z(1-2z^2)\frac{d^2}{dz^2} + (1-6z^2)\frac{d}{dz}\right]$$

from which it follows that

$$\left(\frac{dz^2}{dy}\right)^2 = 4z^4(1-z^2)$$
$$\left(\frac{d^2z^2}{dy^2}\right)^2 = 4z^4(2-3z^2)^2$$
$$\left(\frac{d^3z^2}{dy^3}\right)^2 = 64z^4(1-z^2)(1-3z^2)^2\ .$$

Recalling that $x = y/B$ and multiplying everything out, we get

$$Q_4 = \frac{1}{B}\int_{-\infty}^{\infty} dy\ \left(A^4z^8 - 2Az^2\cdot 4A^2B^2z^4(1-z^2) + \frac{1}{5}A^2B^4 4z^4(2-3z^2)^2\right) =$$
$$= \frac{1}{B}\left(A^4I_4 - 8A^3B^2(I_3-I_4) + \frac{4}{5}A^2B^4(4I_2 - 12I_3 + 9I_4)\right)$$
$$= \frac{32}{105}\frac{3A^2 - 4AB^2 + 2B^4}{B}$$
$$Q_5 = \frac{1}{B}\int_{-\infty}^{\infty} dy\ \Big(A^5z^{10} - 5A^2z^4\cdot 4A^2B^2z^4(1-z^2)$$
$$+ Az^2\cdot A^2B^4 4z^4(2-3z^2)^2 - \frac{32}{7}A^2B^6z^4(1-z^2)(1-3z^2)^2\Big)$$
$$= \frac{1}{B}\Big(A^5I_5 - 20A^4B^2(I_4-I_5) + 4A^3B^4(4I_3 - 12I_4 + 9I_5)$$
$$- \frac{32}{7}A^2B^6(I_2 - 7I_3 + 15I_4 - 9I_5)\Big)$$
$$= \frac{128}{315}\frac{A^2(A-B^2)(2A^2 - 3AB^2 + 3B^4)}{B}$$

At $t = 0$ we use $A = N(N+1)$ and $B = 1$ to evaluate

$$Q_4 = \frac{32}{105}N^2(N+1)^2(3N^4 + 6N^3 - N^2 - 4N + 2)$$
$$Q_5 = \frac{128}{315}N^2(N+1)^2(N^2 + N - 1)(2N^4 + 4N^3 - N^2 - 3N + 3).$$

At $t \to \infty$ we use $A_k = 2k^2$ and $B_k = k$ for the N well-separated solitons ($k = 1, \ldots, N$). Adding up their contributions to the charges we obtain

$$Q_4 = \frac{256}{35} \sum_{k=1}^{N} k^7$$
$$Q_5 = \frac{512}{63} \sum_{k=1}^{N} k^9 \ .$$

Equating the values of the conserved charges at $t = 0$ and $t \to +\infty$ we find the identities

$$\sum_{k=1}^{N} k^7 = \frac{1}{24} N^2(N+1)^2(3N^4 + 6N^3 - N^2 - 4N + 2)$$
$$\sum_{k=1}^{N} k^9 = \frac{1}{20} N^2(N+1)^2(N^2 + N - 1)(2N^4 + 4N^3 - N^2 - 3N + 3) \ .$$

26. (a) Show that the pair of equations

$$(u - v)_+ = \sqrt{2}\, e^{(u+v)/2}$$
$$(u + v)_- = \sqrt{2}\, e^{(u-v)/2}$$

provides a Bäcklund transformation linking solutions of $v_{+-} = 0$ (the wave equation in light-cone coordinates) to those of $u_{+-} = e^u$ (the Liouville equation).

(b) Starting from d'Alembert's general solution $v = f(x^+) + g(x^-)$ of the wave equation, use the Bäcklund transformation from part (a) to obtain the corresponding solutions of the Liouville equation for u. [**Hint**: Set $u(x^+, x^-) = 2U(x^+, x^-) + f(x^+) - g(x^-)$. You might simplify the notation by setting $f(x^+) = \log(F'(x^+))$ and $g(x^-) = -\log(G'(x^-))$, where prime means first derivative.]

Solution

(a) Cross-differentiate the equations to get

$$(u - v)_{+-} = \frac{1}{\sqrt{2}}\, e^{(u+v)/2}(u + v)_- = e^{(u+v)/2} e^{(u-v)/2} = e^u$$
$$(u + v)_{-+} = \frac{1}{\sqrt{2}}\, e^{(u-v)/2}(u - v)_+ = e^{(u-v)/2} e^{(u+v)/2} = e^u \ .$$

Taking sum and difference we obtain

$$u_{+-} = e^u \qquad \text{(Liouville eqn)}$$
$$v_{+-} = 0 \qquad \text{(the wave eqn)}$$

(b) Substituting the general solution $v = f(x^+) + g(x^-)$ of the wave equation in the Bäcklund transform we obtain a system of two first order PDEs for a solution u of the Liouville equation:

$$\begin{cases} u_+ - f'(x^+) = \sqrt{2}\, e^{\left[u+f(x^+)+g(x^-)\right]/2} \\ u_- + g'(x^-) = \sqrt{2}\, e^{\left[u-f(x^+)-g(x^-)\right]/2} \end{cases}$$

The system simplifies if we make the substitution $u(x^+, x^-) = 2U(x^+, x^-) + f(x^+) - g(x^-)$ given in the hint:

$$\begin{cases} 2U_+ = \sqrt{2}\, e^{U+f(x^+)} \\ 2U_- = \sqrt{2}\, e^{U-g(x^-)} \end{cases} \iff \begin{cases} e^{-U}\, U_+ = \frac{1}{\sqrt{2}}\, e^{f(x^+)} \equiv \frac{1}{\sqrt{2}}\, F'(x^+) \\ e^{-U}\, U_- = \frac{1}{\sqrt{2}}\, e^{-g(x^-)} \equiv \frac{1}{\sqrt{2}}\, G'(x^-) \end{cases}$$

where in the last expression we used $f(x^+) = \log(F'(x^+))$ and $g(x^-) = -\log(G'(x^-))$ as suggested. This system can be integrated to get

$$-e^{-U} = \frac{1}{\sqrt{2}} \big(F(x^+) + G(x^-) + c\big)$$

where c is an integration constant. Taking the logarithm of the previous equation,

$$U = -\log\left[-\frac{1}{\sqrt{2}} \big(F(x^+) + G(x^-) + c\big)\right]$$

so

$$\begin{aligned} u &= 2U + \log\left[F'(x^+)G'(x^-)\right] \\ &= -2\log\left[-\frac{1}{\sqrt{2}} \big(F(x^+) + G(x^-) + c\big)\right] + \log\left[F'(x^+)G'(x^-)\right] \\ &= \log \frac{2F'(x^+)G'(x^-)}{\left(F(x^+) + G(x^-) + c\right)^2} \\ &= f(x^+) - g(x^-) - 2\log\big(F(x^+) + G(x^-) + c\big) + \log 2\,. \end{aligned}$$

Remarks:
- since F and G can be arbitrary functions just as much as can be f and g, the second-last line is the most efficient way to write the general solution, while strictly speaking the formula on the last line should be accompanied by the relations between F and f, and G and g;
- given that F and G are arbitrary and adding a constant to either of them does not affect their derivatives, we could take $c = 0$ without losing any generality.

27. Consider the Bäcklund transformation

$$\begin{aligned} &v_x + \frac{1}{2}uv = 0 \\ &v_t + \frac{1}{2}u_x v - \frac{1}{4}u^2 v = 0\,. \end{aligned}$$

(a) Show that these equations taken together imply that v satisfies the linear heat equation $v_t = v_{xx}$, while u satisfies Burgers' equation $u_t + uu_x - u_{xx} = 0$.
[**Hint**: for v, solve the first equation for u and substitute in the second; for u, start by cross-differentiating.]

(b) Find the *general* travelling-wave solution for $v(x,t)$ and, via the Bäcklund transformation, re-obtain the travelling-wave for Burgers' equation found in question 13 (e).

(c) * The linear equation satisfied by $v(x,t)$ allows for the linear superposition of solutions. Use this fact, and your answers to part (b), to construct solutions for v and then u which describe the interaction of *two* travelling waves.

(d) * Sketch your solutions functions of x at fixed times both before and after the interaction, and also draw their trajectories in the (x,t) plane, perhaps starting with the help of a computer. Are the travelling waves of Burgers' equation true solitons, in the sense given in lectures?
[**Hints**: Examine the asymptotics of the solution viewed from frames moving at various velocities V (that is, set $X_V = x - Vt$ and consider $t \to \pm\infty$ keeping X_V finite). This should allow you to isolate various travelling waves in these limits, and to decide whether they preserve their form under interactions. For definiteness, consider the case $c_1 > c_2 > 0$, where c_1 and c_2 are the velocities of the two separate travelling waves before they were superimposed. A further hint: as well as the 'expected' special values for V, namely c_1 and c_2, be careful about what happens when $V = c_1 + c_2$.]

Solution

(a) From the first equation, u can be expressed as $u = -2v_x/v$. Substituting this into the second gives

$$v_t + \frac{1}{2}v\left[-2\frac{v_{xx}}{v} + 4(\frac{v_x}{v})\right] - 2(\frac{v_x}{v})v = 0.$$

This can be rearranged as

$$v_t = v_{xx}\ ,$$

which is the heat equation. A little more elegant is to start by using the first equation to rewrite the second as

$$v_t + \tfrac{1}{2}u_x v + \tfrac{1}{2}uv_x = v_t + \tfrac{1}{2}(uv)_x = 0$$

Now use the first equation again for uv to find $v_t - v_{xx} = 0$, as required.
To get Burgers' equation, first divide both equations by v and use $v'/v = (\log v)'$ to find

$$(\log v)_x + \tfrac{1}{2}u = 0$$
$$(\log v)_t + \frac{1}{2}u_x - \tfrac{1}{4}u^2 = 0\ .$$

Differentiate the first of these with respect to t, the second with respect to x, and subtract:

$$\tfrac{1}{2}u_t - \tfrac{1}{2}u_{xx} + \tfrac{1}{4}(u^2)_x = 0$$

or, multiplying through by 2,

$$u_t - u_{xx} + uu_x = 0$$

which is Burgers' equation.

(b) Substituting $v(x,t) = f(x-ct)$ into the heat equation $v_t = v_{xx}$ (I'm using c instead of v for the velocity of the wave to avoid confusion with the function $v(x,t)$) the equation becomes $-cf' = f'' \Rightarrow -cf = f' + A$ (where A is the constant of integration). Solving,

$$x - x_0 - ct = -\int \frac{df}{cf + A} = -\frac{1}{c}\log\left(f + \frac{A}{c}\right)$$

and so

$$v(x,t) = f(x-ct) = e^{-c(x-x_0-ct)} - \frac{A}{c}\,.$$

Now use the first equation of the Bäcklund transformation to reconstruct $u(x,t)$:

$$\begin{aligned} u(x,t) = -2\frac{v_x}{v} &= -2\frac{\left(-ce^{-c(x-x_0-ct)}\right)}{\left(e^{-c(x-x_0-ct)} - \frac{A}{c}\right)} \\ &= \frac{2c}{1 - \frac{A}{c}e^{c(x-x_0-ct)}}. \end{aligned}$$

In order to ensure a nonsingular solution, we must assume $A/c < 0$. Taking $A = -c$ gives the travelling wave found in question 13 (e).

Remark : A and x_0 are not independent constants in this solution, since shifting A to $A \times \delta$ can be absorbed by a shift in x_0.

(c) Let's set $A/c_i = -1, \;\; i \in \{1,2\}$. The two solutions for $v(x,t)$ are then

$$\begin{aligned} v_1(x,t) &= 1 + e^{-c_1(x-x_1-c_1t)} \\ v_2(x,t) &= 1 + e^{-c_2(x-x_2-c_2t)} \end{aligned}$$

with x_1, x_2, c_1, c_2 all constants. The equation for v being linear, $v_1 + v_2$ is also a solution. Via the first equation of the BT this gives

$$\begin{aligned} u(x,t) = -2\frac{(v_{1x} + v_{2x})}{v_1 + v_2} &= \frac{-2\left(-c_1e^{-c_1(x-x_1-c_1t)} - c_2e^{-c_2(x-x_2-c_2t)}\right)}{\left(2 + e^{-c_1(x-x_1-c_1t)} + e^{-c_2(x-x_2-c_2t)}\right)} \\ &= \frac{2\left(c_1e^{-c_1(x-x_1-c_1t)} + c_2e^{-c_2(x-x_2-c_2t)}\right)}{\left(2 + e^{-c_1(x-x_1-c_1t)} + e^{-c_2(x-x_2-c_2t)}\right)} \end{aligned}$$

(d) An example solution is shown in figure 1, where $c_1 = 2.5$, $c_2 = 1$, and $x_1 = x_2 = 0$. Looking at the figure it's clear that the two separate waves visible for $t \to -\infty$ merge into a single wave as $t \to +\infty$. The waves therefore do not retain their form

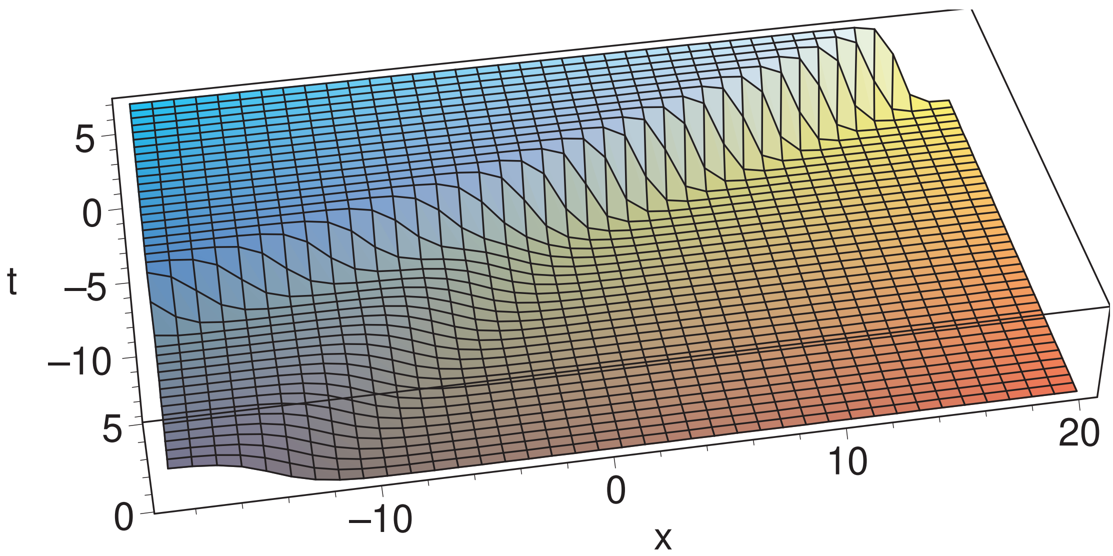


Figure 1: Two-wave solution of Burgers' equation $u_t + uu_x - u_{xx} = 0$

in collisions, and so they are not solitons in the sense given in lectures – they fail to satisfy property 3.

For a more careful analysis, the tactic will be as in lectures: switch to a frame of reference moving with some velocity c by setting $X_c = x - ct$, and examine the form of $u(X_c, t)$ as $t \to \pm\infty$ with X_c remaining finite. Supposing that $x_1 = x_2 = 0$, u as a function of X_c and t is

$$u(X_c, t) = 2\,\frac{c_1 e^{-c_1(X_c+(c-c_1)t)} + c_2 e^{-c_2(X_c+(c-c_2)t)}}{2 + e^{-c_1(X_c+(c-c_1)t)} + e^{-c_2(X_c+(c-c_2)t)}}\,.$$

Keeping X_c finite corresponds to following the solution at speed c. As $t \to \pm\infty$, we expect to observe different behaviours depending on the value of c relative to c_1 , c_2 and (following the hint) $c_1 + c_2$. We'll assume that $c_1 > c_2$, and define

$$\begin{aligned}\theta_1 &= c_1(x - c_1 t) = c_1 X_c + c_1(c - c_1)t\,,\\ \theta_2 &= c_2(x - c_2 t) = c_2 X_c + c_2(c - c_2)t\end{aligned}$$

so that

$$u(X_c, t) = 2\,\frac{c_1 e^{-\theta_1} + c_2 e^{-\theta_2}}{2 + e^{-\theta_1} + e^{-\theta_2}}\,.$$

The limits work much as for sine-Gordon. The trick is to rearrange the formula for u when taking each limit so as to avoid having ∞/∞...

• $t \to +\infty$

i. $c > c_1$: $\theta_1 \to +\infty$ and $\theta_2 \to +\infty$, so $e^{-\theta_1} \to 0$ and $e^{-\theta_2} \to 0$ and $u \to 0$.

ii. $c = c_1$: $\theta_2 \to +\infty$ while θ_1 remains finite, so $u \to \frac{2c_1 e^{-\theta_1}}{2+e^{-\theta_1}} = \frac{2c_1}{1+2e^{\theta_1}}$, a simple Burger's equation travelling wave with velocity c_1.

iii. $c_1 > c > c_2$: $\theta_1 \to -\infty$ and $\theta_2 \to +\infty$, so $e^{-\theta_1} \to \infty$ and $e^{-\theta_2} \to 0$, and $u = 2\,\frac{c_1 + c_2 e^{\theta_1} e^{-\theta_2}}{2e^{\theta_1} + 1 + e^{\theta_1} e^{-\theta_2}} \to 2c_1$.

iv. $c = c_2$: $\theta_1 \to -\infty$ and θ_2 remains finite, so $e^{-\theta_1} \to \infty$ and $e^{-\theta_2}$ remains finite, and so $u = 2\,\frac{c_1+c_2 e^{\theta_1}e^{-\theta_2}}{2e^{\theta_1}+1+e^{\theta_1}e^{-\theta_2}} \to 2c_1$.

v. $c < c_2$: $\theta_1 \to -\infty$ and $\theta_2 \to -\infty$, so $e^{-\theta_1} \to \infty$ and $e^{-\theta_2} \to \infty$ and, since $c_1 > c_2$, $e^{\theta_1}e^{-\theta_2} \to 0$. Hence $u = 2\,\frac{c_1+c_2 e^{\theta_1}e^{-\theta_2}}{2e^{\theta_1}+1+e^{\theta_1}e^{-\theta_2}} \to 2c_1$.

• $t \to -\infty$ (in slightly less detail)

i. $c > c_1$: $\theta_1 \to -\infty$ and $\theta_2 \to -\infty$, and there are three subcases:

A. $c > c_1 + c_2$: $\theta_1 - \theta_2 \to -\infty$ and $u \to 2c_1$.

B. $c = c_1 + c_2$: $\theta_1 - \theta_2$ remains finite and $u \to 2\frac{c_1 e^{-(c_1-c_2)X_c}+c_2}{e^{-(c_1-c_2)X_c}+1}$.

C. $c < c_1 + c_2$: $\theta_1 - \theta_2 \to +\infty$ and $u \to 2c_2$.

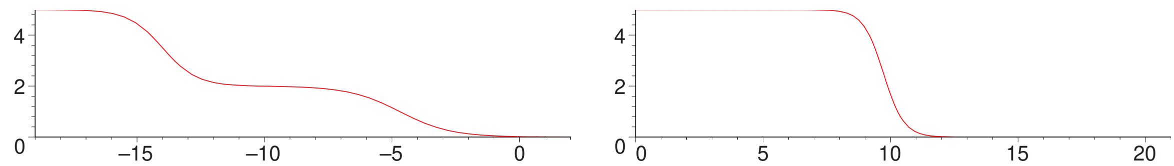
ii. $c = c_1$: $\theta_2 \to -\infty$ while θ_1 remains finite, so $u \to 2c_2$.

iii. $c_1 > c > c_2$: $\theta_1 \to +\infty$ and $\theta_2 \to -\infty$, so $u \to 2c_2$.

iv. $c_1 > c = c_2$: $\theta_1 \to +\infty$ and θ_2 remains finite, so $u \to \frac{2c_2}{1+2e^{c_2 X_c}}$, a simple travelling wave with velocity c_2 .

v. $c < c_2$: $\theta_1 \to +\infty$ and $\theta_2 \to +\infty$, then $u \to 0$.

Putting these results together allows the solution to be sketched in the two limits – the results are shown in figure 2.



$t = -4$ $t = 4$

Figure 2: The solution for $c_1 = 2.5$ and $c_2 = 1$ at two different times

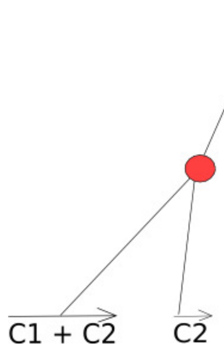


Figure 3: Burgers' equation: a faster wave "eating" a slower one

The form of the waves is <u>not</u> preserved under collision (the faster wave "eats" the slower one) and so these travelling waves are not true solitons (see figure 3).

Note: this question in its entirety is much *more involved than would be asked in an exam, where you would also be given more hints as to which steps to take. But it would still be good to make sure that, with the solution to hand, you understand how it goes.*

28. (a) Show that the two equations

$$v_x = -u - v^2$$
$$v_t = 2u^2 + 2uv^2 + u_{xx} - 2u_x v$$

are a Bäcklund transformation relating solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

and the wrong sign modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0\,.$$

(Note the appearance of the Miura transform in the Bäcklund transformation.)

(b) Taking $u = c^2$, where c is a constant, as a seed solution of the KdV equation, find the corresponding solution of the wrong sign mKdV equation.

Solution

(a) We'll label the two equations of the BT as follows:

$$v_x = -u - v^2 \qquad \text{(a)}$$
$$v_t = 2u^2 + 2uv^2 + u_{xx} - 2u_x v \qquad \text{(b)}$$

Solving (a) for u, $u = -v^2 - v_x$. Substituting into (b),

$$v_t = 2(v^2 + v_x)^2 - 2(v^2 + v_x)v^2 - (v^2 + v_x)_{xx} + 2(v^2 + v_x)_x v\,.$$

After some cancellations, the RHS simplifies to $6v^2 v_x - v_{xxx}$, so v satisfies mKdV. For u, take $\frac{\partial(\mathrm{a})}{\partial t}$ and use (b) for v_t to find

$$v_{xt} = -u_t - 2vv_t = -u_t - 2v(2u^2 + 2uv^2 + u_{xx} - 2u_x v) \qquad \text{(c)}$$

Next, take $\frac{\partial(\mathrm{b})}{\partial x}$ and use (a) for v_x to find

$$\begin{aligned} v_{tx} &= 4uu_x + 2u_x v^2 + 4uvv_x + u_{xxx} - 2u_{xx}v - 2u_x v_x \\ &= 4uu_x + 2u_x v^2 + u_{xxx} - 2u_{xx}v + (4uv - 2u_x)(-u - v^2) \end{aligned} \qquad \text{(d)}$$

Since $v_{tx} = v_{xt}$, the last lines of (c) and (d) are equal. All terms involving v cancel, leaving $-u_t = 6uu_x + u_{xxx}$, the KdV equation.

(b) Substituting $u = c^2$ into the BT leads to

$$v_x = -c^2 - v^2 \qquad \text{(e)}$$
$$v_t = 2c^4 + 2c^2 v^2 \qquad \text{(f)}$$

Solving (e),

$$x = -\int \frac{dv}{c^2 + v^2} = -\frac{1}{c}\arctan(v/c) + f(t) \quad \Rightarrow \quad \arctan(v/c) = -c(x - f(t))$$

where $f(t)$ is the x-independent constant of integration. Similarly from (f)

$$t = \frac{1}{2c^2}\int \frac{dv}{c^2+v^2} = \frac{1}{2c^3}\arctan(v/c)+g(x) \quad \Rightarrow \quad \arctan(v/c) = 2c^3(t-g(x))$$

Equating the two, $-c(x-f(t)) = 2c^3(t-g(x))$, or $x-2c^2g(x) = -2c^2t+f(t) =$ const (since LHS and RHS depend on x and t only) $= x_0$, say. Hence $f(t) = x_0 + 2c^2t$ and

$$\arctan(v/c) = -c(x-x_0-2c^2t) \quad \Rightarrow \quad v(x,t) = -c\tan\big(c(x-x_0-2c^2t)\big)\,.$$

(For c real, the solution is singular. That sometimes happens with Bäcklund transformations, but a nonsingular solution can be found if we allow c to be purely imaginary.)
A quick check: (e) and (f) together imply $(\frac{\partial}{\partial t}+2c^2\frac{\partial}{\partial x})v(x,t)=0$, which is clearly satisfied by the final answer.

29. The 2-soliton solution of the sine-Gordon equation with Bäcklund parameters a_1 and a_2 is

$$u(x,t) = 4\arctan\left(\mu\,\frac{e^{\theta_1}-e^{\theta_2}}{1+e^{\theta_1+\theta_2}}\right)\,, \qquad \theta_i = \varepsilon_i\gamma_i(x-v_it-\bar{x}_i)$$

where $\mu = (a_2+a_1)/(a_2-a_1)$, $v_i = (a_i^2-1)/(a_i^2+1)$, $\gamma_i = 1/\sqrt{1-v_i^2}$, $\varepsilon_i = \text{sign}(a_i)$, and $\bar{x}_1$ and $\bar{x}_2$ are constants, as in the lectures. Rewriting u as a function of $X_V \equiv x - Vt$ and t, show that, for $V \neq v_1, v_2$ (and $v_1 \neq v_2$)

$$\lim_{\substack{t\to\infty \\ X_V \text{ finite}}} u = 2n\pi\,,$$

where n is an integer. If $v_2 > v_1 > 0$ and $\varepsilon_i = 1$, how does the parity of n (whether it is even or odd) depend on the value of v relative to v_1 and v_2?
[**Hints**: First show that $|\theta_i| \to +\infty$ as $t \to \pm\infty$; then consider each of the four possible options $(\theta_1,\theta_2) \to (+\infty,+\infty)$, $(-\infty,-\infty)$, $(+\infty,-\infty)$, $(-\infty,+\infty)$. Remember that $\arctan(0) = m\pi$ and $\arctan(\pm\infty) = \pm\pi/2+m\pi$, where the ambiguities of $m\pi$, $m \in \mathbb{Z}$, encode the multivalued nature of the arctan function.]

Solution Working with (X_v,t) instead of (x,t):

$$\theta_i = \varepsilon_i\gamma_i(X_v+(v-v_i)t)-\bar{x}_i)$$

As in lectures, if $v \neq v_1$ and $v \neq v_2$ then $v-v_1 \neq 0$ and $v-v_2 \neq 0$ so both $|\theta_1|$ and $|\theta_2|$ tend to ∞ as $t \to \infty$. (In fact, $\theta_i \to \text{sign}(\varepsilon_i(v-v_i)\times\infty\,.)$ It's probably easiest just to look at the various cases in turn:

(a) $(\theta_1,\theta_2) \to (+\infty,+\infty)$: then $e^{\theta_i} \to +\infty$ and $e^{-\theta_i} \to 0$ for $i=1,2$ and

$$\mu\,\frac{e^{\theta_1}-e^{\theta_2}}{1+e^{\theta_1+\theta_2}} = \mu\,\frac{e^{-\theta_2}-e^{-\theta_1}}{e^{-\theta_1-\theta_2}+1} \to \mu\,\frac{0-0}{0+1} = 0$$

The first equality comes on dividing top and bottom by $e^{\theta_1+\theta_2}$.

(b) $(\theta_1, \theta_2) \to (-\infty, -\infty)$: then e^{θ_1} and e^{θ_2} both tend to zero, and so

$$\mu \frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}} \to \mu \frac{0 - 0}{1 + 0} = 0$$

(c) $(\theta_1, \theta_2) \to (+\infty, -\infty)$: then $e^{\theta_1} \to \infty$ while $e^{\theta_2} \to 0$ and so

$$\mu \frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}} \sim \mu \frac{e^{\theta_1}}{1 + e^{\theta_1+\theta_2}} = \mu \frac{1}{e^{-\theta_1} + e^{\theta_2}} \to \mathrm{sign}(\mu) \times \infty$$

Here the first step relies on the fact that e^{θ_1} wins over e^{θ_2} in the first numerator, while the final one uses $e^{-\theta_1} \to 0$, $e^{\theta_2} \to 0$.

(d) $(\theta_1, \theta_2) \to (-\infty, +\infty)$: this is just like case **(c)** apart from a minus sign, so the limit is $-\mathrm{sign}(\mu) \times \infty$.

Taking the $4 \times \arctan$ of these limits and using the values of $\arctan$ that the question is kind enough to quote for you, it follows that for cases **(a)** and **(b)** the limit of u as $t \to \infty$ with X_v finite is $4m\pi$, while in case **(c)** it is $2\pi\,\mathrm{sign}(\mu) + 4m\pi$, and in case **(d)**, $-2\pi\,\mathrm{sign}(\mu) + 4m\pi$. Since $\mathrm{sign}(\mu) = \pm 1$, in every case the limit is equal to $2n\pi$ for some integer n ; and furthermore n is even in cases **(a)** and **(b)**, and odd in cases **(c)** and **(d)**.

To complete the final part, we need to check which of cases **(a)**, **(b)**, **(c)** or **(d)** hold for the various options for v when $v_2 > v_1 > 0$ and $\varepsilon_1 = \varepsilon_2 = 1$. The given values imply $a_2 > a_1 > 0$, and so $\mu > 0$, $\mathrm{sign}(\mu) = +1$, and $\theta_i \to \mathrm{sign}(v - v_i) \times \infty$. Thus the options are:

- $v < v_1 < v_2$: $(\theta_1, \theta_2) \to (-\infty, -\infty)$ – case **(b)**, n even;
- $v_1 < v < v_2$: $(\theta_1, \theta_2) \to (+\infty, -\infty)$ – case **(c)**, n odd;
- $v_1 < v_2 < v$: $(\theta_1, \theta_2) \to (+\infty, +\infty)$ – case **(a)**, n even.

It's not a bad idea to compare these predictions with the plots on the course webpage, just to see that they check out.

30. Find the asymptotics of the 2-soliton sine-Gordon solution defined in problem 29, in the case $a_2 > a_1 > 0$, as $t \to \pm\infty$ with $X_{v_2} \equiv x - v_2 t$ held finite.

    Solution This is solved in the same way as the case covered in lectures. However because we are now following the faster of the two solitons, the limits work a little differently and you should recover the profile of a single antikink rather than the kink that you might have expected to find, with a total forwards phase shift of $\frac{2}{\gamma_2} \log((a_2 + a_1)/(a_2 - a_1))$.

31. Show by direct analysis (as in the lectures) that taking a_1 and a_2 of opposite signs in problem 29 results in a two-kink, or two-antikink, solution to the sine-Gordon equation.

Solution Again, this is just a case of grinding through the various cases, tracking the solution at various speeds and paying particular attention when those speeds are equal to v_1 or v_2.

32. (a) The argument of the arctangent in the sine-Gordon 2-soliton solution of problem 29 is a continuous function of x for all $x \in \mathbb{R}$. In particular, it is never infinite. What does this imply about the range of u? [**Hint**: consider the graph of $\tan u/4$.]
    (b) By taking the limits of this function as $x \to \pm\infty$ (with $t = \bar{x}_1 = \bar{x}_2 = 0$ for simplicity), show that the topological charge of this two-soliton solution is 0 if $\text{sign}(a_1) = \text{sign}(a_2)$, and ± 2 if $\text{sign}(a_1) = -\text{sign}(a_2)$, in units where the topological charge of a kink is 1.

Solution

(a) The argument of the arctangent in the sine-Gordon 2-soliton solution is equal to $\mu(e^{\theta_1} - e^{\theta_2})/(1 + e^{\theta_1+\theta_2})$. Since θ_1 and θ_2 are both real, $e^{\theta_1+\theta_2}$ is positive for all x, and so the denominator is never zero, and hence the whole function is continuous and never infinite. Hence $\tan(u/4)$ is never infinite, which means that $u/4$ is never equal to $\pm\pi/2$. Thus if u is in the range $(-2\pi, 2\pi)$ for any value of x, it must remain in that range for all other values of x.

(b) For $t = \bar{x}_1 = \bar{x}_2 = 0$ the argument of the arctangent is

$$\mu\left(\frac{e^{\epsilon_1\gamma_1 x} - e^{\epsilon_2\gamma_2 x}}{1 + e^{\epsilon_1\gamma_1 x + \epsilon_2\gamma_2 x}}\right)$$

where $\epsilon_i = \text{sign}(a_i)$. If $\epsilon_1 = \epsilon_2 = 1$ or $\epsilon_1 = \epsilon_2 = -1$ the limit of this function as $x \to \pm\infty$ is zero. Picking the branch of the arctangent such that u tends to 0 as $x \to -\infty$, the limit as $x \to \infty$ must be an integer multiple of 4π; but by the first part of the question, u must stay in the range $(-2\pi, 2\pi)$ and hence the limit as $x \to \infty$ of u must also be zero, and so the topological charge of u is 0. (Note that this is consistent with the earlier calculations which showed that for $a_2 > a_1 > 1$ the solution (3) consists of a kink and an antikink.) The calculation is similar for $\text{sign}(a_1) = -\text{sign}(a_2)$.
Note: this question is filling in a detail from the lectures, but I wouldn't expect you to reproduce the whole argument in an exam.

33. Consider the two-soliton solution of the sine-Gordon equation from problem 29 with complex Bäcklund parameters $a_1 = a_2^* := a \in \mathbf{C}$ and with vanishing integration constants, as is appropriate to find the breather solution. Show that

$$\text{Re}(\theta_1) = +\text{Re}(\theta_2) = \gamma(x - vt)\cos\varphi\ ,$$
$$\text{Im}(\theta_1) = -\text{Im}(\theta_2) = \gamma(vx - t)\sin\varphi\ ,$$

where $\varphi = \arg(a)$ and

$$v = \frac{|a|^2 - 1}{|a|^2 + 1}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1 + |a|^2}{2|a|}\ .$$

Solution This is largely bookwork – have a look at section 6.8 of the printed notes (but note that this material is only examinable for those on the 1-year MSc course).

34. The *stationary* breather solution of the sine-Gordon equation (that is the breather solution with $v = 0$) has the form

$$\tan\frac{u}{4} = \frac{\cos\varphi}{\sin\varphi}\cdot\frac{\sin(t\sin\varphi)}{\cosh(x\cos\varphi)}\ .$$

Show that in the limit $\varphi \to 0$, in which the kink and antikink that form the breather are very loosely bound, the time period τ of a single oscillation of the breather scales like $\tau \sim |\varphi|^{-1}$, and the spatial size $x_{\max}$ of the breather scales like $x_{\max} \sim -\log\varphi$.
[**Hint**: You could define $x_{\max}$ as the value of x at which $\tan(u/4) = 1$ when the oscillatory factor in the numerator is at its maximum. Focus only on the parametric dependence on φ, ignoring all numerical factors.]

Solution I'll leave this for you to think about, but please feel free to ask me about it.

35. We have seen in lectures that the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$ that describes the profile of a wave translates into the following equation for the new variable $w(x,t) = \int dx\ u$:

$$w_t + 3w_x^2 + w_{xxx} = 0\ .$$

Let $w = 2\frac{\partial}{\partial x}\log f = 2f_x/f$ where $f(x,t)$ is a nowhere vanishing function of x and t, so that $u = 2\frac{\partial^2}{\partial x^2}\log f$. The aim of this exercise is to rewrite the equation for w as an equation for f.

(a) Express w_t, w_x, w_{xx} and w_{xxx} in terms of f and its derivatives.

(b) Show that the equation for $w_t + 3w_x^2 + w_{xxx} = 0$ can be rewritten as

$$ff_{xt} - f_xf_t + 3f_{xx}^2 - 4f_xf_{xxx} + ff_{xxxx} = 0\ ,$$

which is known as the *quadratic form* of the KdV equation.

Solution This is a case of hacking through the equations to fill in the (small) gaps in the derivation given in the printed notes. It's a good exercise to try it for yourself.

36. The Hirota bilinear differential operator $D_t^mD_x^n$ is defined for any pair of natural numbers (m,n) by

$$D_t^mD_x^n(f\cdot g) = \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x\\ t'=t}}$$

and maps a pair of functions $(f(x,t), g(x,t))$ into a single function.

(a) Prove that the Hirota operators $B_{m,n} := D_t^mD_x^n$ are bilinear, *i.e.* for all constants a_1, a_2

$$B_{m,n}(a_1f_1 + a_2f_2\cdot g) = a_1B_{m,n}(f_1\cdot g) + a_2B_{m,n}(f_2\cdot g)\ ,$$
$$B_{m,n}(f\cdot a_1g_1 + a_2g_2) = a_1B_{m,n}(f\cdot g_1) + a_2B_{m,n}(f\cdot g_2)\ .$$

(b) Prove the symmetry property

$$B_{m,n}(f \cdot g) = (-1)^{m+n} B_{m,n}(g \cdot f)\ .$$

(c) Compute the Hirota derivatives $D_t^2(f \cdot g)$ and $D_x^4(f \cdot g)$, and verify that your expression for the latter is consistent with the result for $D_x^4(f \cdot f)$ given in lectures.

Solution

(a) This follows fairly directly from the definitions – for a complete proof you can expand out all terms using the binomial theorem and the use the linearity of all of the partial derivatives involved.

(b) As in part (a), this can be proved quickly starting from the defnition of Hirota's bilinear operator - but it is worth checking you understand how it goes.

(c)

$$\begin{aligned} D_t^2(f \cdot g) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^2 f(x,t)g(x',t')\Big|_{\substack{x'=x\\t'=t}} \\ &= \left(f_{tt}g - 2f_t g_{t'} + f g_{t't'}\right)\Big|_{\substack{x'=x\\t'=t}} \\ &= f_{tt}g - 2f_t g_t + f g_{tt} \end{aligned}$$

$$\begin{aligned} D_x^4(f \cdot g) &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^4 f(x,t)g(x',t')\Big|_{\substack{x'=x\\t'=t}} \\ &= \left(\frac{\partial^4}{\partial x^4} - 4\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial x'} + 6\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial x'^2} - 4\frac{\partial}{\partial x}\frac{\partial^3}{\partial x'^3} + \frac{\partial^4}{\partial x'^4}\right) f(x,t)g(x',t')\Big|_{\substack{x'=x\\t'=t}} \\ &= f_{xxxx}g - 4f_{xxx}g_x + 6f_{xx}g_{xx} - 4f_x g_{xxx} + f g_{xxxx} \end{aligned}$$

Taking $f = g$ gives $D_x^4(f \cdot f) = 2(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2)$, as required.

37. Define a "not-Hirota" bilinear differential operator $\tilde{D}_t^m \tilde{D}_x^n$ by

$$\tilde{D}_t^m \tilde{D}_x^n(f \cdot g) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x'=x\\t'=t}}$$

(note the plus signs!).

(a) Compute $\tilde{D}_x(f \cdot g)$ and $\tilde{D}_t(f \cdot g)$, verifying that in both cases the answer is given by the corresponding 'ordinary' derivative of the product $f(x,t)g(x,t)$.

(b) How does this result generalise for arbitrary not-Hirota differential operators? Prove your claim.

(c) Compare your answer with the Hirota operators defined above.

Solution

(a) For the first case:

$$\begin{aligned}\tilde{D}_x(f\cdot g) &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right) f(x,t)g(x',t')\Big|_{\substack{x'=x\\ t'=t}}\\ &= (f_x g + f g_{x'})\Big|_{\substack{x'=x\\ t'=t}} = f_x g + f g_x = \frac{\partial}{\partial x}(fg)\,.\end{aligned}$$

In just the same way, $\tilde{D}_t(f\cdot g)=\frac{\partial}{\partial t}(fg)$.

(b) Generalisation:

$$\tilde{D}_t^m\tilde{D}_x^n(f\cdot g)=\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n}\left(f(x,t)g(x,t)\right)$$

Proof: expand out both sides, using the binomial formula for the $n^{\text{th}}/m^{\text{th}}$ derivative of a product on the RHS.

38. (a) If $\theta_i = a_i x + b_i t + c_i$, prove that

$$D_tD_x(e^{\theta_1}\cdot e^{\theta_2})=(b_1-b_2)(a_1-a_2)e^{\theta_1+\theta_2}.$$

(b) Prove the corresponding result for $D_t^mD_x^n(e^{\theta_1}\cdot e^{\theta_2})$, as quoted in lectures.

Solution

(a) We have

$$\begin{aligned}D_tD_x(e^{\theta_1}\cdot e^{\theta_2}) &= \left(\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)e^{a_1x+b_1t+c_1}e^{a_2x'+b_2t'+c_2}\right)\Bigg|_{\substack{x'=x\\ t'=t}}\\ &= \left(\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)(a_1-a_2)\,e^{a_1x+b_1t+c_1}e^{a_2x'+b_2t'+c_2}\right)\Bigg|_{\substack{x'=x\\ t'=t}}\\ &= \left((b_1-b_2)\,(a_1-a_2)\,e^{a_1x+b_1t+c_1}e^{a_2x'+b_2t'+c_2}\right)\Big|_{\substack{x'=x\\ t'=t}}\\ &= (b_1-b_2)\,(a_1-a_2)\,e^{\theta_1x+\theta_2}\end{aligned}$$

as required.

(b) To save space, write $\theta_1 = a_1x+b_1t+c_1$, and $\theta_2' = a_2x'+b_2t'+c_2$. Then note that $\frac{\partial}{\partial t}e^{\theta_1}e^{\theta_2'}=b_1e^{\theta_1}e^{\theta_2'}$; $\frac{\partial}{\partial t'}e^{\theta_1}e^{\theta_2'}=b_2e^{\theta_1}e^{\theta_2'}$, etc, so

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m e^{\theta_1}e^{\theta_2'} &= \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^{m-1}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)e^{\theta_1}e^{\theta_2'}\\ &= \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^{m-1}(b_1-b_2)\,e^{\theta_1}e^{\theta_2'}\\ &= \ldots = (b_1-b_2)^m e^{\theta_1}e^{\theta_2'}\,.\end{aligned}$$

Similarly

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n e^{\theta_1}e^{\theta'_2} \;=\; (a_1-a_2)^n(b_1-b_2)^m e^{\theta_1}e^{\theta'_2}\,.$$

Putting these results together,

$$\begin{aligned} D_t^m D_x^n(e^{\theta_1}\cdot e^{\theta_2}) &= \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n e^{\theta_1}e^{\theta'_2}\Big|_{\substack{x'=x\\ t'=t}} \\ &= (b_1-b_2)^m\,(a_1-a_2)^n\,e^{\theta_1}e^{\theta'_2}\Big|_{\substack{x'=x\\ t'=t}} \\ &= (b_1-b_2)^m\,(a_1-a_2)^n\,e^{\theta_1}e^{\theta_2}\,. \end{aligned}$$

39. Prove that

$$D_t^m D_x^n(f\cdot 1) = \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n}f\,.$$

Solution

$$D_t^m D_x^n(f\cdot 1) = \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n f(x,t)\Big|_{\substack{x'=x\\ t'=t}}$$

Now note,

$$\begin{aligned} \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n f(x,t) &= \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^{n-1}\left(\frac{\partial}{\partial x}f(x,t)-\frac{\partial}{\partial x'}f(x,t)\right) \\ &= \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^{n-1}\frac{\partial}{\partial x}f(x,t) \;=\; \dots \;=\; \frac{\partial^n}{\partial x^n}f(x,t)\,. \end{aligned}$$

(Don't forget that $\frac{\partial}{\partial x'}f(x,t)=0$.) The same holds for $\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m$, so the result follows. For $D_t^m D_x^n(1\cdot f)$, you get an extra factor of $(-1)^m$ from $\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m$, and $(-1)^n$ from $\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n$.

40. Consider the function f, such that $u=2\frac{\partial^2}{\partial x^2}\log f$ is the KdV field, which corresponds to a 2-soliton solution:

$$f = 1+\epsilon f_1+\epsilon^2 f_2 = 1+\epsilon\left(e^{\theta_1}+e^{\theta_2}\right)+\epsilon^2\left(\frac{a_1-a_2}{a_1+a_2}\right)^2 e^{\theta_1+\theta_2}\,,$$

where $\theta_i=a_ix-a_i^3t+c_i$, with a_i and c_i constants. Check that $B(f_1\cdot f_2)=0$ and $B(f_2\cdot f_2)=0$, where $B=D_x(D_t+D_x^3)$, and show that this implies that the above expansion, which is truncated at order ϵ^2, is a solution of the bilinear form of the KdV equation.

Solution Here's part of the answer: equation A_3 is the $n=3$ case of equation (7.20) from notes, which is

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial t^3}\right)f_n = -\frac{1}{2}\sum_{m=1}^{n-1}B(f_{n-m}\cdot f_m)\,. \tag{7.20}$$

Up to a factor of $-1/2$ the RHS of (7.20) for $n = 3$ is $B(f_1 \cdot f_2) + B(f_2 \cdot f_1)$, as stated in the question. Substituting in and dropping another overall factor, this time $(a_1 - a_2)^2/(a_1 + a_2)^2$, $B(f_1 \cdot f_2) + B(f_2 \cdot f_1)$ is proportional to

$$B(e^{\theta_1} \cdot e^{\theta_1+\theta_2}) + B(e^{\theta_2} \cdot e^{\theta_1+\theta_2}) + B(e^{\theta_1+\theta_2} \cdot e^{\theta_1}) + B(e^{\theta_1+\theta_2} \cdot e^{\theta_2})\,, \qquad (*)$$

where $\theta_1 = a_1x - a_1^3t + c_1$, $\theta_2 = a_2x - a_2^3t + c_2$, $\theta_1 + \theta_2 = (a_1+a_2)x - (a_1^3+a_2^3)t + c_1+c_2$. Now from lemma 1 in your notes, if $\vartheta_1 = \alpha_1 x + \beta_1 t + \gamma_1$, $\vartheta_2 = \alpha_1 x + \beta_2 t + \gamma_2$, then

$$D_t^m D_x^n (e^{\vartheta_1} \cdot e^{\vartheta_2}) = (\beta_1 - \beta_2)^m (\alpha_1 - \alpha_2)^n e^{\vartheta_1+\vartheta_2}\,. \qquad (7.13)$$

Since $B = D_t D_x + D_x^4$, this implies

$$\begin{aligned} B(e^{\theta_1} \cdot e^{\theta_1+\theta_2}) &= \left((-a_1^3 + a_1^3 + a_2^3)(a_1 - a_1 - a_2) + (a_1 - a_1 - a_2)^4\right) e^{2\theta_1+\theta_2} \\ &= (a_2^3(-a_2) + (-a_2)^4) e^{2\theta_1+\theta_2} \;=\; 0 \end{aligned}$$

and likewise for the other three terms, so $B(f_1 \cdot f_2) + B(f_2 \cdot f_1) = 0$ as required. It's easy to check that all further terms f_4, f_5, ... can also be set to zero, which means that the potentially-infinite series $f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \ldots$ does indeed terminate at order ϵ^2 for this choice of f_1. Setting $\epsilon = 1$ gives the exact KdV solution $u(x,t) = 2\frac{\partial^2}{\partial x^2}\log(1 + f_1 + f_2)$.

41. Derive the solution of the bilinear form of the KdV equation $D_x(D_t + D_x^3)(f \cdot f) = 0$ which represents the 3-soliton solution, in the form

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3$$

where $f_1 = \sum_{i=1}^3 e^{\theta_i}$. [This includes proving that the higher order terms in the ϵ expansion can be consistently set to zero, as in problem 40.]

Solution This is quite a long task! But the final answer turns out to fit the general formula given at the end of section 7.3 of your notes.

42. Show that the Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0$$

can be written in the bilinear form

$$(D_t^2 - D_x^2 - D_x^4)(f \cdot f) = 0$$

where $u = 2\frac{\partial^2}{\partial x^2}\log f$.

Solution Substituting in $u = 2\frac{\partial^2}{\partial x^2}\log f$ and integrating twice in x,

$$2(\log f)_{tt} - 2(\log f)_{xx} - 3 \cdot 4(\log f)_{xx}^2 - 2(\log f)_{xxxx} = a(t)x + b(t)$$

where $a(t)$ and $b(t)$ are the constants (in x) of integration. We're free to redefne $\log f \longrightarrow \log f + \alpha(t)x + \beta(t)$ without changing u. Picking α and β such that $2\alpha' i(t)' = a(t)$ and $2\beta''(t) = b(t)$ we can absorb the RHS, and the equation becomes

$$2(\log f)_{tt} - 2(\log f)_{xx} - 3 \cdot 4(\log f)_{xx}^2 - 2(\log f)_{xxxx} = 0\,.$$

The terms are

$$\begin{aligned}
2(\log f)_{tt} &= 2\,\frac{f_{tt}f - f_t^2}{f^2} = \frac{D_t^2(f\cdot f)}{f^2}\,,\\
2(\log f)_{xx} &= 2\,\frac{f_{xx}f - f_x^2}{f^2} = \frac{D_x^2(f\cdot f)}{f^2}\,,\\
\Rightarrow 3\cdot 4(\log f)_{xx}^2 &= 12\,\frac{f_{xx}^2 f^2 - 2f_{xx}ff_x^2 + f_x^4}{f^4}\,,\\
2(\log f)_{xxxx} &= \dots\\
&= 2\,\frac{-6f_x^4 + 12ff_x^2f_{xx} - 3f^2f_{xx}^2 - 4f^2f_xf_{xxx} + f^3f_{xxxx}}{f^4}\,.
\end{aligned}$$

Adding the last twp equations,

$$3\cdot 4(\log f)_{xx}^2 + 2(\log f)_{xxxx} = 2\,\frac{iff_{xxxx} - 4f_{xxx}f_x + 3f_{xx}^2}{f^2} = \frac{D_x^4(f\cdot f)}{f^2}\,.$$

Finally, combining everything and multiplying by f^2 yields the Boussinesq equation in bilineaar form:

$$\left(D_t^2 - D_x^2 - D_x^4\right)(f\cdot f) = 0\,.$$

43. Show that the following higher-dimensional version of the KdV equation,

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$$

for the field $u(x, y, t)$, also known as the Kadomtsev-Petviashvili (KP) equation, can be written in the bilinear form

$$(D_tD_x + D_x^4 + 3\sigma^2 D_y^2)(f\cdot f) = 0$$

where $u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log f(x, y, t)$.

Solution This is left for you to try – but you can check your answer against that given in the question.

44. It is given that the system of Hirota equations

$$\begin{cases}(D_x^2 - D_t^2 - 1)(f\cdot g) = 0\\ (D_x^2 - D_t^2)(f\cdot f) = (D_x^2 - D_t^2)(g\cdot g)\end{cases}$$

yields solutions $u = 4\arctan(g/f)$ of the sine-Gordon equation. Let $\theta_i = a_ix + b_it + c_i$, where a_i, b_i, c_i are constants.

(a) Take

$$f = 1\,, \qquad g = \epsilon e^{\theta_1}$$

and work order by order in powers of ϵ to find the one-soliton solution of the sine-Gordon equation.

(b) Taking e^{θ_i} as in the solution of the previous part, repeat the exercise for

$$f = 1 + \epsilon^2 f_2\,, \qquad g = \epsilon(e^{\theta_1} + e^{\theta_2})\,,$$

and check that the Hirota equations are satisfied to all orders in ϵ.

Solution

(a) Let's write the system of Hirota equations as a power series in ϵ, using the shorthand $B = D_x^2 - D_t^2$ and bilinearity:

$$\begin{cases} 0 = (B-1)(f \cdot g) = (B-1)(1\cdot 0) + \epsilon(B-1)(1 \cdot e^{\theta_1}) \\ 0 = B(1\cdot 1) - \epsilon^2 B(e^{\theta_1} \cdot e^{\theta_1}) \end{cases}$$

Now we solve the system order by order.

- Order ϵ^0: we have the system

$$\begin{cases} 0 = (B-1)(1 \cdot 0) \\ 0 = B(1\cdot 1) \end{cases} .$$

  This is trivially satisfied, since $B(1 \cdot 0) = 1(1\cdot 0) = B(1\cdot 1) = 0$, using bilinearity and/or direct differentiation.

- Order ϵ^1: we have the equation

$$0 = (B-1)(1 \cdot e^{\theta_1}) = (\partial_x^2 - \partial_t^2 - 1)e^{\theta_1} = (a_1^2 - b_1^2 - 1)e^{\theta_1}\,,$$

  so the parameters must satisfy $a_1^2 = b_1^2 + 1$.

- Order ϵ^2: we have the equation $B(e^{\theta_1} \cdot e^{\theta_1}) = 0$, which is trivially satisfied using Lemma 1 from the notes (or explicit differentiation).

(b) Subbing in

$$f = 1 + \epsilon^2 f_2\,, \qquad g = \epsilon g_1 \equiv \epsilon(e^{\theta_1} + e^{\theta_2}),$$

using some of the above results, bilinearity and the symmetry property of $B(h \cdot k) = B(k \cdot h)$, we find the system

$$\begin{cases} 0 = (B-1)((1+\epsilon^2 f_2)\cdot(\epsilon g_1)) = \epsilon(B-1)(1\cdot g_1) + \epsilon^3(B-1)(f_2 \cdot g_1) \\ 0 = B((1+\epsilon^2 f_2)\cdot(1+\epsilon^2 f_2)) - B((\epsilon g_1)\cdot(\epsilon g_1)) = \epsilon^2\left[2B(f_2\cdot 1) - B(g_1 \cdot g_1)\right] + \epsilon^4 B(f_2 \cdot f_2) \end{cases}$$

We need to solve the system order by order. Note that we have a single equation at each order, since the first Hirota equation is odd in ϵ, while the second Hirota equation is even.

- Order ϵ^0: this is trivially satisfied.
- Order ϵ^1: the equation $(B-1)(1\cdot(e^{\theta_1}+e^{\theta_2}))=0$ is satisfied using bilinearity and $a_i^2=b_i^2+1$ (as in part a).
- Order ϵ^2: we have the equation

$$\begin{aligned} 0 &= 2B(f_2\cdot 1) - B((e^{\theta_1}+e^{\theta_2})\cdot(e^{\theta_1}+e^{\theta_2})) \\ &= 2(\partial_x^2-\partial_t^2)f_2 - 2B(e^{\theta_1}\cdot e^{\theta_2}) \\ &= 2(\partial_x^2-\partial_t^2)f_2 - 2\left[(a_2-a_1)^2-(b_2-b_1)^2\right]e^{\theta_1+\theta_2}\ , \end{aligned}$$

  using bilinearity and Lemmata 1 and 2. (It's understood that $a_i^2=b_i^2+1$.) This equation determines f_2. As for the 2-soliton solution of KdV, we can take $f_2=Ae^{\theta_1+\theta_2}$ for a constant A to be determined. Subbing in the previous equation, we obtain

$$A\left[(a_1+a_2)^2-(b_1+b_2)^2\right]e^{\theta_1+\theta_2} = \left[(a_2-a_1)^2-(b_2-b_1)^2\right]e^{\theta_1+\theta_2}\ ,$$

  which determines
$$A=\frac{(a_2-a_1)^2-(b_2-b_1)^2}{(a_1+a_2)^2-(b_1+b_2)^2}\ .$$
  One can use $a_i^2=b_i^2+1$ to simplify the result, but this is not necessary.
- Order ϵ^3: simplifying a factor of A, we have to check that the equation

$$0=(B-1)(e^{\theta_1+\theta_2}\cdot(e^{\theta_1}+e^{\theta_2}))=(B-1)(e^{\theta_1+\theta_2}\cdot e^{\theta_1})+(B-1)(e^{\theta_1+\theta_2}\cdot e^{\theta_2})$$

  is satisfied, since there are no unknowns left. The two terms in the RHS vanish individually, as in the solution to problem 40. Let's check it explicitly for the first term (for the second term, swap 1 and 2):

$$\left[(a_1+a_2-a_1)^2-(b_1+b_2-b_1)^2-1\right]e^{2\theta_1+\theta_2}=[a_2^2-b_2^2-1]e^{2\theta_1+\theta_2}=0$$

  using $a_i^2=b_i^2+1$.
- Order ϵ^4: we have

$$B(f_2,f_2)=A^2B(e^{\theta_1+\theta_2}\cdot e^{\theta_1+\theta_2})=0$$

  by Lemma 1 or by explicit differentiation.

45. **Note:** In this and subsequent exercises the Fourier transform will be denoted as $\mathbf{F}[f(x)] = \widetilde{f}(k)$, where $\mathbf{F}[f(x)] = \widetilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\widetilde{f}(k)$. You can use results from the Fourier transform handout such as $\delta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{iyz}$ without proof.

Some properties of Fourier transforms:

(a) The *convolution* of f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} dz\ f(z)\, g(x-z)\ .$$

Prove that $\mathbf{F}[fg] = \frac{1}{2\pi}\widetilde{f}(k) * \widetilde{g}(k)$ and $\mathbf{F}[f * g] = \widetilde{f}(k)\widetilde{g}(k)$.

(b) The *cross-correlation* of f and g is defined as

$$(f \otimes g)(x) = \int_{-\infty}^{\infty} dz\ f^*(z)\, g(x+z)\ .$$

Prove the *Weiner-Kinchin theorem,* that $\mathbf{F}[f \otimes g] = \widetilde{f}^*(k)\widetilde{g}(k)$.

(c) The *auto-correlation* of $f(x)$ is defined as

$$a(x) = (f \otimes f)(x).$$

Using the answer to part b, verify that $\mathbf{F}[a] = |\widetilde{f}(k)|^2$. This is called the *energy spectrum* of f.

(d) Prove the FT version of *Parseval's theorem*, which you may have already seen for Fourier series:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, |\widetilde{f}(k)|^2\ .$$

(Strictly speaking this is *Plancherel's theorem*; Parseval allows for two different functions f and g and turns into Plancherel when $f = g$.)

The locations of the factors of 2π in these formulae depend on the conventions used for the Fourier transform and its inverse, so they might look a little different in some textbooks.

Solution To save space all integrals will henceforth be assumed to run from $-\infty$ to ∞ unless otherwise stated.

(a) Convolution: $(f * g)(x) = \int dz\ f(z)\, g(x-z)$.
We have

$$\begin{aligned}
\mathbf{F}[fg] &= \int dx\, e^{-ikx} f(x) g(x) \\
&= \int dx \iint \frac{dk_1 dk_2}{(2\pi)^2}\, e^{-i(k-k_1-k_2)x}\, \widetilde{f}(k_1)\widetilde{g}(k_2) \\
&= \iint \frac{dk_1 dk_2}{2\pi}\, \delta(k-k_1-k_2)\widetilde{f}(k_1)\widetilde{g}(k_2) \quad \text{(doing the } \textstyle\int dx) \\
&= \frac{1}{2\pi}\int dk_1\, \widetilde{f}(k_1)\widetilde{g}(k-k_1) = \frac{1}{2\pi}(\widetilde{f} * \widetilde{g})(k)\,.
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}[f*g] &= \int dx\, e^{-ikx} \int dz\, f(z)g(x-z) \\
&= \int dx\, e^{-ikx} \int dz \iint \frac{dk_1 dk_2}{(2\pi)^2}\, e^{ik_1 z + ik_2(x-z)} \widetilde{f}(k_1)\widetilde{g}(k_2) \\
&= \int dz \iint \frac{dk_1 dk_2}{2\pi}\, e^{i(k_1-k_2)z} \delta(k_2-k)\widetilde{f}(k_1)\widetilde{g}(k_2) \quad \text{(doing the } \textstyle\int dx) \\
&= \iint dk_1 dk_2\, \delta(k_1-k_2)\delta(k_2-k)\widetilde{f}(k_1)\widetilde{g}(k_2) \quad \text{(doing the } \textstyle\int dz) \\
&= \widetilde{f}(k)\widetilde{g}(k)\,.
\end{aligned}$$

(b) Cross-correlation: $(f \otimes g)(x) = \int dz\, f^*(z)\, g(x+z)$.
Note that $(f \otimes g)(x) = \int dy\, f^*(-y)\, g(x-y)$, which is the convolution of $f^*(-x)$ and $g(x)$. So if we can show that $\mathbf{F}[f^*(-x)](k) = \widetilde{f}^*(k)$ it will follow from part (a) that $\mathbf{F}[f \otimes g] = \widetilde{f}^*(k)\widetilde{g}(k)$. This holds since

$$\begin{aligned}
\mathbf{F}[f^*(-x)](k) &= \int dx\, f^*(-x)e^{-ikx} \\
&= \left(\int dx\, f(-x)e^{ikx}\right)^* \\
&= \left(\int dy\, f(y)e^{-iky}\right)^* \quad \text{(substituting } y=-x) \\
&= \widetilde{f}^*(k)
\end{aligned}$$

(c) Auto-correlation: $a(x) = (f \otimes f)(x)$.
Using part (b), $\mathbf{F}[a] = \mathbf{F}[f \otimes f] = \widetilde{f}^*(k)\widetilde{f}(k) = |\widetilde{f}(k)|^2$.

(d) Parseval's theorem: $\int dx\, |f(x)|^2 = \frac{1}{2\pi} \int dk\, |\widetilde{f}(k)|^2$.

$$\begin{aligned}
\int dx\, |f(x)|^2 &= \int dx \iint \frac{dk_1 dk_2}{(2\pi)^2}\, e^{-i(k_1-k_2)x} \widetilde{f}^*(k_1)\widetilde{f}(k_2) \\
&= \iint \frac{dk_1 dk_2}{2\pi}\, \delta(k_1-k_2)\widetilde{f}^*(k_1)\widetilde{f}(k_2) \quad \text{(doing the } \textstyle\int dz) \\
&= \int \frac{dk_1}{2\pi}\, \widetilde{f}^*(k_1)\widetilde{f}(k_1) = \int \frac{dk}{2\pi}\, |\widetilde{f}(k)|^2\,.
\end{aligned}$$

46. Examples of Fourier transforms:

(a) Show that $e^{-x^2/2}$ is (up to a factor of $\sqrt{2\pi}$) its own FT.

(b) Find the FT of

$$f(x) = \begin{cases} 1/(2\varepsilon) & |x| \le \varepsilon \\ 0 & |x| > \varepsilon \end{cases}$$

and discuss the $\varepsilon \to 0$ limit.

(c) Find the FT of

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases} .$$

Solution

(a)

$$\begin{aligned}
\mathbf{F}[e^{-x^2/2}](k) &= \int dx\, e^{-ikx} e^{-x^2/2} \\
&= \int dx\, e^{-(x^2+2ikx)/2} \\
&= \int dx\, e^{-((x+ik)^2+k^2)/2} \quad \text{(completing the square)} \\
&= e^{-k^2/2} \int dx\, e^{-(x+ik)^2/2} \\
&= e^{-k^2/2} \int dx\, e^{-x^2/2} \quad \text{(shifting } x \to x + ik \text{ as on the integrals sheet)} \\
&= \sqrt{2\pi}\, e^{-k^2/2} \quad \text{(using the definite integral from the integrals sheet).}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{F}[f(x)](k) &= \int_{-\epsilon}^{\epsilon} dx\, e^{-ikx} \frac{1}{2\epsilon} \\
&= \left[e^{-ikx} \frac{-1}{2ik\epsilon} \right]_{-\epsilon}^{\epsilon} \\
&= \frac{1}{\epsilon k} \sin(k\epsilon)
\end{aligned}$$

As $\epsilon \to 0$ with k fixed this tends to 1, a constant. Now consider the Fourier transform of a Dirac delta function: $\mathbf{F}[\delta(x)](k) = \int dx\, e^{-ikx}\delta(x) = 1$ – it's the same! If you think about the shape of the original function $f(x)$ in the limit, this might seem reasonable.

(c) In this case $\mathbf{F}[f(x)](k) = \int_{-1}^{1} dx\, e^{-ikx}(1 - x^2)$. As a shortcut which avoids integrating by parts, define

$$I(k) = \int_{-1}^{1} dx\, e^{-ikx} = \frac{1}{-ik} \left[e^{-ikx} \right]_{-1}^{1} = \frac{2}{k} \sin(k)$$

and notice, differentiating inside the integral for the first equality, that

$$\frac{d^2}{dk^2} I(k) = -\int_{-1}^{1} dx\, x^2 e^{-ikx} = \frac{d^2}{dk^2} \left(\frac{2}{k} \sin(k) \right) = \frac{4}{k^3} \sin(k) - \frac{4}{k^2} \cos(k) - \frac{2}{k} \sin(k) .$$

Thus

$$\mathbf{F}[f(x)](k) = \left(I(k) + \frac{d^2}{dk^2} I(k) \right) = \frac{4}{k^3} \left(\sin(k) - k \cos(k) \right) .$$

47. Solving the heat equation using Fourier transforms:

(a) Find the general solution of the heat equation $u_t = u_{xx}$ in the form

$$u(x,t) = \int_{-\infty}^{+\infty} dk\, \widetilde{u}(k,0) f(k,x,t) \,,$$

where $\widetilde{u}(k,0)$ is the Fourier transform of the initial condition $u(x,0)$ and $f(k,x,t)$ is a function of k, x and t that you should determine.

(b) Evaluate the previous integral over k in the case where the initial condition is $u(x,0) = \delta(x)$, to obtain the corresponding solution $u(x,t)$ for $t > 0$ explicitly. [**Hint**: look at the definite integrals on the useful integrals sheet and read the note below.]

(c) Finally, derive the general solution as in equation (7.2) in the lecture notes.

Solution

(a) Taking the Fourier transform of the heat equation and integrating by parts twice on the u_{xx} term, $\widetilde{u}(k,t)$ must solve

$$\widetilde{u}_t + k^2 \widetilde{u} = 0$$

which is a first-order ODE, easily solved for any value of k:

$$\widetilde{u}(k,t) = \widetilde{u}(k,0)\, e^{-k^2 t} \,.$$

Transforming back,

$$u(x,t) = \frac{1}{2\pi} \int dk\, \widetilde{u}(k,t)\, e^{ikx} = \frac{1}{2\pi} \int dk\, \widetilde{u}(k,0)\, e^{-k^2 t + ikx} \,.$$

(b) For $u(x,0) = \delta(x)$ it's easy to compute that $\widetilde{u}(k,0) = 1$. Substituting this into the result from part (a),

$$u(x,t) = \frac{1}{2\pi} \int dk\, e^{-k^2 t + ikx} = \frac{1}{2\pi} \int dk\, e^{-t(k - i\frac{x}{2t})^2} e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}}$$

using the Gaussian integral from the integrals sheet for the definite integral, after shifting the integration variable by a finite imaginary amount as in the note below the integral.

(c) In the general case we want $u(x,0) = u_0(x)$, so (using x' instead of x for the integration variable in the FT) $\widetilde{u}(k,0) = \int dx' e^{-ikx'} u_0(x')$. Inserting this into the formula found in part (a) and then doing the k integral just as in part (b), though with x replaced by $x - x'$,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \iint dk\, dx'\, e^{-ikx'} u_0(x')\, e^{-k^2 t + ikx} \\ &= \frac{1}{2\pi} \iint dk\, dx'\, u_0(x')\, e^{-k^2 t + ik(x - x')} \\ &= \frac{1}{2\sqrt{\pi t}} \int dx'\, u_0(x')\, e^{-\frac{(x-x')^2}{4t}} \end{aligned}$$

which is indeed formula (7.2) from lectures. Note, you can think of this as "adding up" (using an integral) lots of solutions to the problem from part (b) – this is the motivation for the idea of a *Green's function*.

48. Find the general solution of the linearised KdV equation $u_t + u_{xxx} = 0$. Your answer should be in the form of an integral involving $\widetilde{u}(k, 0)$, the Fourier transform of the initial condition $u(x, 0)$.

Solution Taking the Fourier transform of the linearised KdV equation $u_t + u_{xxx} = 0$,

$$\widetilde{u}_t = ik^3\widetilde{u}$$

which has the solution $\widetilde{u}(k,t) = e^{ik^3t}\widetilde{u}(k,0)$. Transforming back,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{u}(k,t)\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik^3t}\widetilde{u}(k,0)\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(k^2t+x)}\widetilde{u}(k,0)\, dk \end{aligned}$$

where $\widetilde{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx$. Note that this is a superposition of waves travelling leftwards, in line with numerical simulations of small-amplitude waves in the full KdV equation.

49. Try to solve the full (non-linear) KdV equation using the same method, Fourier transform. [Do not try too hard as it is impossible! Just convince yourself that it is impossible and understand what goes wrong/why the Fourier transform doesn't work in the non-linear case.]

50. Show that if $u(x,t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where λ is a constant and $v(x,t)$ some other function, then v satisfies

$$\big(2v + \frac{\partial}{\partial x}\big)\big(v_t + 6(\lambda - v^2)v_x + v_{xxx}\big) = 0\,.$$

(You might recognise this problem from last term!)

Solution Differentiating $u = \lambda - v^2 - v_x$ yields:

$$\begin{aligned} u_t &= -2vv_t - v_{tx} \\ u_x &= -2vv_x - v_{xx} \\ u_{xx} &= -2v_x^2 - 2vv_{xx} - v_{xxx} \\ u_{xxx} &= -6v_xv_{xx} - 2vv_{xxx} - v_{xxxx}\,. \end{aligned}$$

Substituting into the KdV equation, and noting that $(v^2v_x)_x = v^2v_{xx} + 2vv_x^2$, we find

$$-2v\left[v_t + 6\lambda v_x - 6v^2v_x + v_{xxx}\right] - \frac{\partial}{\partial x}\left[v_t + 6\lambda v_x - 6v^2v_x + v_{xxx}\right] = 0,$$

and thus

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t + 6\lambda v_x - 6v^2 v_x + v_{xxx}\right) = 0\,.$$

51. If λ is an eigenvalue of $\frac{d^2}{dx^2}\psi(x)+u(x)\psi(x) = \lambda\psi(x)$, where we require that $\int_{-\infty}^{\infty}|\psi(x)|^2 dx < \infty$, and $u(x)$ is real, prove that λ must also be real. [**Hint:** start by multiplying by $\psi(x)^*$ and integrating.]

Solution Following the hint, we find

$$\int_{-\infty}^{\infty} dx \left[\psi(x)^* \frac{d^2}{dx^2}\psi(x) + \psi(x)^* u(x)\psi(x)\right] = \lambda \int_{-\infty}^{\infty} dx\ |\psi(x)|^2$$

and integrating the first term on the LHS by parts once,

$$\int_{-\infty}^{\infty} dx \left[-|d\psi(x)/dx|^2 + u(x)|\psi(x)|^2\right] = \lambda \int_{-\infty}^{\infty} dx\ |\psi(x)|^2\ .$$

Since $u(x)$ is real, the LHS is real; and dividing through by $\int_{-\infty}^{\infty} dx\ |\psi(x)|^2$ (which is finite, real and nonzero) shows that λ is real.

52. The *Wronskian* $W[f,g](x)$ of two differentiable functions $f(x)$ and $g(x)$ is defined as

$$W[f,g](x) = f'(x)g(x) - f(x)g'(x)\ .$$

If the functions f and g are linearly dependent, then their Wronskian vanishes identically: $W[f,g](x) = 0$. (Equivalently, if $W[f,g](x) \neq 0$, the functions f and g are linearly independent.) Conversely, if the Wronskian vanishes identically for two *analytic* functions f and g, then f and g are linearly dependent.

(a) Write down the Wronskian $W[\psi_1^*, \psi_2](x)$ of two eigenfunctions $\psi_{1,2}(x)$ of the time-independent Schrödinger equation with the same potential $V(x)$ and possibly different eigenvalues k_i^2:

$$\psi_i''(x) - V(x)\psi_i(x) = -k_i^2\psi_i(x) \qquad (i = 1, 2)\ . \qquad (**)$$

(This is just preparation for what follows, no computation is needed.)

(b) Show that the Wronskian is constant if the two eigenfunctions correspond to the same eigenvalue.

(c) Show that two eigenfunctions with different eigenvalues are orthogonal with respect to the (hermitian) inner product

$$(\psi_1, \psi_2) := \int_{-\infty}^{+\infty} dx\ \psi_1^*(x)\psi_2(x)$$

if at least one of the two eigenfunctions describes a bound state.

(d) Show that the Wronskian vanishes for two eigenfunctions with the same eigenvalue in the discrete spectrum. (This implies the linear dependence of the two eigenfunctions, provided that they are analytic.) [**Hint**: consider the limit $x \to \pm\infty$.]

(e) The $x \to \pm\infty$ asymptotics of a scattering solution $\psi(x)$ with eigenvalue $k^2 > 0$ is

$$\psi(x) \approx \begin{cases} e^{ikx} + R(k)\, e^{-ikx}\,, & x \to -\infty \\ T(k)\, e^{ikx}\,, & x \to +\infty \end{cases}$$

By evaluating the Wronskian $W[\psi^*, \psi]$ at $x \to \pm\infty$, show that the reflection and transmission coefficients $R(k)$ and $T(k)$ satisfy

$$|R(k)|^2 + |T(k)|^2 = 1\,.$$

Solution

(a) $W[\psi_1^*, \psi_2] = \psi_1^{*\prime}(x)\psi_2(x) - \psi_1^*(x)\psi_2'(x)$.

(b) We have

$$\frac{d}{dx}W[\psi_1^*, \psi_2](x) = \psi_1^{*\prime\prime}\psi_2 + \psi_1^{*\prime}\psi_2' - \psi_1^{*\prime}\psi_2' - \psi_1^*\psi_2'' = \psi_1^{*\prime\prime}\psi_2 - \psi_1^*\psi_2''\,.$$

Then use the differential equation (details should be given) to substitute for $\psi_1^{*\prime\prime}$ and ψ_2'' to see that the two terms on the RHS cancel when the eigenvalues are the same, so the Wronskian is indeed constant.

(c) Multiply the complex conjugate of (**) with $i = 1$ by $\psi_2(x)$ and subtract $\psi_1(x)^*$ times (**) with $i = 2$ and integrate from $-\infty$ to ∞ to find

$$\int_{-\infty}^{\infty} (\psi_1^{*\prime\prime}\psi_2 - \psi_1^*\psi_2'')\, dx = (k_2^2 - k_1^2)\,(\psi_1, \psi_2)\,.$$

As in the solution to part 2, the integrand on the LHS of this equation is equal to $\frac{d}{dx}W[\psi_1^*, \psi_2](x)$, and integrates to zero since at least one of ψ_1 and ψ_2 is a bound state and vanishes at $\pm\infty$ (while the other, even if not in the discrete spectrum, must be bounded at infinity). Hence for $k_1^2 \neq k_2^2$, $(\psi_1, \psi_2) = 0$.

(d) From part 2, the Wronskian of the two eigenfunctions, sharing the same eigenvalue, is constant. Since this eigenvalue is in the discrete spectrum these eigenfunctions vanish at $\pm\infty$, and so their Wronskian vanishes there. It therefore vanishes for all x, as required.

(e) As $x \to +\infty$,

$$W[\psi^*, \psi] \to (T(k)^*e^{-ikx})'\, T(k)e^{ikx} - T(k)^*e^{-ikx}\,(T(k)e^{ikx})' = -2ik|T(k)|^2\,.$$

Likewise, as $x \to -\infty$

$$\begin{aligned} W[\psi^*, \psi] &\to (e^{-ikx} + R(k)^*e^{ikx})'\,(e^{ikx} + R(k)e^{-ikx}) \\ &\quad - (e^{-ikx} + R(k)^*e^{ikx})\,(e^{ikx} + R(k)e^{-ikx})' \\ &= -2ik + 2ik|R(k)|^2 = -2ik(1 - |R(k)|^2)\,. \end{aligned}$$

Since by part 2 this Wronskian is constant, these two limits must agree, and with a little rearrangement the desired result follows.

53. Consider the time independent Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi(x)=k^2\psi(x)$$

with energy $E=k^2$ for the square barrier/well potential

$$V(x)=\begin{cases}0\,, & x<0\\ V_0\,, & 0<x<a\\ 0\,, & x>a\end{cases}$$

where $a>0$ and V_0 are constants.

(a) Show that the matching conditions to be imposed at $x=0$ and a, where the square well potential is discontinuous (but finite), are that $\psi(x)$ and $\psi'(x)$ are continuous.

(b) Solve the Schrödinger equation for this potential in the three given regions and impose the matching conditions to find the scattering solutions associated to energy eigenvalues $k^2>0$ in the continuous spectrum, and determine the reflection and transmission coefficients $R(k)$ and $T(k)$ in terms of a and $l=\sqrt{k^2-V_0}$.

(c) For which values of the wavenumber k is the square well potential transparent, that is $R(k)=0$?

(d) Write down the bound state solutions corresponding to the discrete spectrum $k^2=-\mu^2<0$. Find the equations that determine implicitly the allowed values of μ in terms of a and l (or V_0).

(e) Do bound state solutions exist for $V_0>0$? And for $V_0<0$? In the latter case, use a graphical argument to show that a new bound state solution appears every time that $\sqrt{-V_0}$ crosses a non-negative integer multiple of π/a.

(f) Show that in the limit $a\to 0$, $V_0\to+\infty$ with $b=aV_0$ fixed, the reflection and transmission coefficients reduce to those of the delta-function potential $V(x)=b\delta(x)$.

Solution See the handwritten notes for the problems class.

54. Consider the time independent Schrödinger equation

$$-\psi''(x)+V(x)\psi(x)=k^2\psi(x)\,,$$

where the potential $V(x)$ is the sum of two delta functions:

$$V(x)=-a\delta(x)-b\delta(x-r)\,.$$

Taking $r>0$, the solution $\psi(x)$ can be split into three pieces, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$, defined on $(-\infty,0)$, $(0,r)$, and $(r,+\infty)$ respectively.

(a) Write down the four matching conditions relating ψ_1, ψ_2 and ψ_3, and their derivatives, at $x=0$ and $x=r$.

(b) For a scattering solution describing waves incident from the left, ψ_1 and ψ_3 are given by

$$\psi_1(x) = e^{ikx} + R(k)\, e^{-ikx}, \quad \psi_3(x) = T(k)\, e^{ikx}.$$

Write down the general form of ψ_2, and then use the matching conditions found in part 1 to eliminate the unknowns and determine $R(k)$ and $T(k)$.

(c) Show from the answer to part 2 that, for there to be a bound state pole at $k = i\mu$, μ must satisfy

$$e^{-2\mu r} = (1 - 2\mu/a)(1 - 2\mu/b)\ . \qquad (***)$$

(d) The solutions to (***) can be analysed using a graphical method. Show that:

i. if both a and b are negative, then there are no bound states;
ii. if a and b have opposite signs, then there is at most one bound state, occurring when $a + b > rab$ (note: since a and b have opposite signs, rab is negative);
iii. if a and b are positive, then the number of bound states is one if $rab \leq a + b$, and two otherwise.

Sketch on the ab-plane the regions which correspond to zero, one and two bound states, and indicate the form of $\psi(x)$ for each of the two bound states found when $ab/(a + b) > r^{-1}$.

Solution We have

$$\psi(x) = \begin{cases} \psi_1(x)\,, & x < 0\,, \\ \psi_2(x)\,, & 0 < x < r\,, \\ \psi_3(x)\,, & x > r\,. \end{cases}$$

(a) With $\psi = \psi_1$, ψ_2 or ψ_3 as above, at $x = 0$ we have

$$\psi(0^-) = \psi(0^+) \equiv \psi(0)\ , \quad \psi'(0^+) - \psi'(0^-) = -a\psi(0)\ ,$$

while at $x = r$,

$$\psi(r^-) = \psi(r^+) \equiv \psi(r)\ , \quad \psi'(r^+) - \psi'(r^-) = -b\psi(r)\ .$$

(b)

$$\psi(x) = \begin{cases} e^{ikx} + R(k)\, e^{-ikx}\,, & x < 0\,, \\ A(k)\, e^{ikx} + B(k)(k)\, e^{-ikx}\,, & 0 < x < r\,, \\ T(k)\, e^{ikx}\,, & x > r\,. \end{cases}$$

Imposing the matching conditions at $x = 0$,

$$1 + R(k) = A(k) + B(k)\ , \quad ik(A(k) - B(k)) - ik(1 - R(k)) = -a(1 + R(k))\,,$$

so

$$A(k) + B(k) = 1 + R(k)\,, \qquad (\alpha)$$

$$A(k) - B(k) = \left(1 + i\frac{a}{k}\right) - R(k)\left(1 - i\frac{a}{k}\right)\ . \qquad (\beta)$$

Likewise, looking at $x = r$,

$$A(k)\,e^{ikr} + B(k)\,e^{-ikr} = T(k)\,e^{ikr}$$
$$ikT(k)\,e^{ikr} - ik(A(k)\,e^{ikr} - B(k)\,e^{-ikr}) = -bT(k)\,e^{ikr},$$

so

$$A(k)\,e^{ikr} + B(k)\,e^{-ikr} = T(k)\,e^{ikr} \qquad (\gamma)$$
$$A(k)\,e^{ikr} - B(k)\,e^{-ikr} = T(k)\,e^{ikr}\left(1 - i\frac{b}{k}\right). \qquad (\delta)$$

Solving these for $A(k)$ and $B(k)$,

$$(\gamma) + (\delta): \;\; A(k) = \left(1 - i\frac{b}{2k}\right) T(k)\,; \quad (\gamma) - (\delta): \;\; B(k) = i\frac{b}{2k}\,e^{2ikr}\,T(k)\,.$$

Thus (α) and (β) become:

$$1 + R(k) = \left(1 - i\frac{b}{2k} + i\frac{b}{2k}\,e^{2ikr}\right) T(k) \qquad (\alpha')$$

$$\left(1 - i\frac{b}{2k} - i\frac{b}{2k}\,e^{2ikr}\right) T(k) = \cdots = 2 - \left(1 - i\frac{a}{k}\right)(1 + R(k)) \qquad (\beta')$$

and substituting for $1 + R(k)$ from (α') into (β') and solving for $T(k)$ yields

$$T(k) = \frac{4k^2/(ab)}{e^{2ikr} - (1 + i\frac{2k}{a})(1 + i\frac{2k}{b})} = \frac{4k^2}{abe^{2ikr} - (a + 2ik)(b + 2ik)}\,.$$

Finally we can use (α') once more to find

$$R(k) = \frac{1 + i\frac{2k}{b} - (1 - i\frac{2k}{a})e^{2ikr}}{e^{2ikr} - (1 + i\frac{2k}{a})(1 + i\frac{2k}{b})} = \frac{a(b + 2ik) - b(a - 2ik)e^{2ikr}}{abe^{2ikr} - (a + 2ik)(b + 2ik)}\,.$$

NOTE: problems like this can be solved more systematically using 'transfer matrices'. Ask me about them if you are interested.

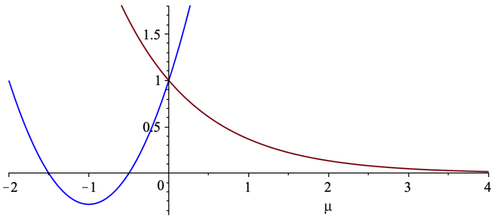
(c) Bound states occur at poles in $T(k)$ with $k = i\mu$, $\mu > 0$. This needs the denominator of the above formula for $T(k)|_{k=i\mu}$ to vanish, that is

$$e^{-2\mu r} = \left(1 - \frac{2\mu}{a}\right)\left(1 - \frac{2\mu}{b}\right)$$

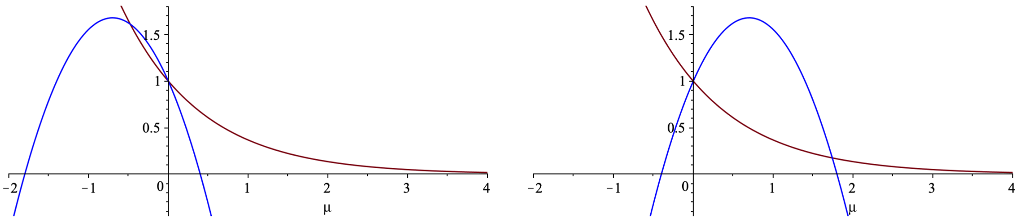
as required.

(d) The LHS of (***), plotted in red below, is a simple decaying exponential, while the RHS (plotted in blue) is a quadratic in μ with zeros at $\mu = a/2$ and $\mu = b/2$. The two curves always intersect at $\mu = 0$; bound states will occur if there are further intersections with $\mu > 0$. Going case by case,

(a) For $a < 0$, $b < 0$, both zeros of the RHS are negative and so there are no intersections with $\mu > 0$:



(b) When a and b have opposite signs, there is one negative and one positive zero of the RHS, and the number of intersections with $\mu > 0$ will be either zero or one:



Which one occurs depends on the relative gradients of the LHS and RHS at $\mu = 0$. These gradients are

$$G_L = \frac{d}{d\mu} e^{-2\mu r} \bigg|_{\mu=0} = -2r$$

and

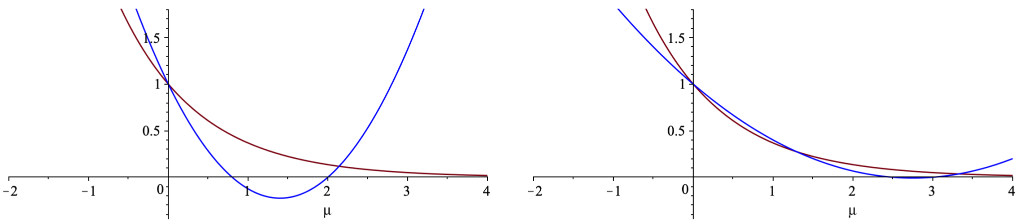
$$G_R = \frac{d}{d\mu} \left(1 - \frac{2\mu}{a}\right)\left(1 - \frac{2\mu}{b}\right)\bigg|_{\mu=0} = -\frac{2}{a} - \frac{2}{b} = -2\frac{(a+b)}{ab}$$

and we are in the situation of the right-hand plot, with one bound state, when $G_L < G_R$, ie $-2r < -2(a+b)/(ab)$, or $r > (a+b)/(ab)$, or (noting that $ab < 0$ when rearranging the inequality)

$$a + b > rab\,,$$

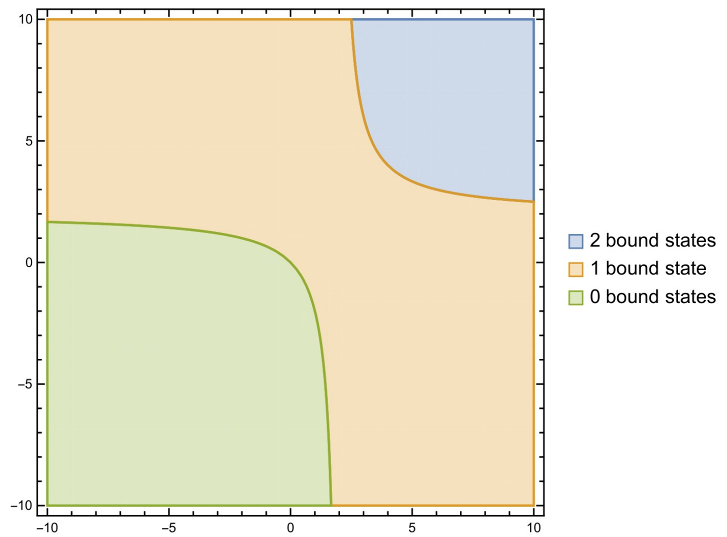
as required. Note that this should indeed be a strict inequality: when $a + b = rab$ the gradients at the origin are equal, and by considering the second derivatives (or otherwise) it can be shown that the only intersection is at $\mu = 0$, which does not give a bound state.

(c) When a and b are both positive, both zeros of the RHS are positive, and the number of intersections with $\mu > 0$ is either one or two:

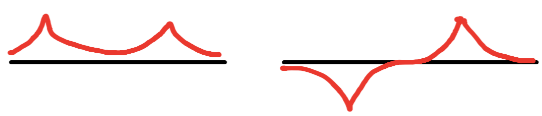


Again, a comparison of the derivatives of the two curves at $\mu = 0$ determines which situation arises, and calculating as above shows that there is one bound state for $rab \leq a + b$ and two otherwise. Also as above, extra arguments need to be made when $rab = a + b$ to get the right answer in this case too.

For the last part, note that the transitions in the numbers of bound states occur on the curves $rab = a + b$, or $rab - a - b = 0$, or $r(a - 1/r)(b - 1/r) = 1/r$. On the a, b plane this is the hyperbola $b = 1/a$, but with the asymptotes shifted up and to the right, to $b = 1/r$ and $a = 1/r$. Here's a region plot in the (a, b)-plane for $r = 1/2$:



Finally, here's a rough sketch of the forms that $\psi(x)$ takes in the zone where there are two bound states:



55. The time independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)$$

is conjectured to have solutions in the form

$$\psi(x) = e^{ikx}(2k + iw(x)) \ ,$$

where $w(x)$ is real, non-singular for all x, independent of k, and has finite limits as $x \to \pm\infty$. Substituting in, deduce the equation

$$w'(x) + \frac{1}{2}w^2(x) = 2\mu^2 \ ,$$

where μ is an integration constant. [**Hint**: take real and imaginary parts of an intermediate equation.] Solve this via the substitution $w(x) = 2f'(x)/f(x)$, and deduce that $V(x)$ must have the form

$$V(x) = -2\mu^2 \operatorname{sech}^2(\mu(x - x_0)) .$$

Show also that $u = -V$ is a solution of the KdV equation provided that x_0 depends on t in a certain way that you should determine.

Solution Substituting in, we need

$$\begin{aligned}
0 &= \left(-\frac{d^2}{dx^2} + V(x) - k^2\right)\left(e^{ikx}(2k + iw(x))\right) \\
&= e^{ikx}\left(2k^3 + ik^2 w(x) + 2kw'(x) - iw''(x) + 2kV(x) + iw(x)V(x) - 2k^3 - ik^2 w(x)\right) \\
&= e^{ikx}\left(2kw'(x) - iw''(x) + 2kV(x) + iw(x)V(x)\right) \\
&= e^{ikx}\left(2k(V(x) + w'(x)) + i(w(x)V(x) - w''(x))\right).
\end{aligned}$$

Setting real and imaginary parts of the term in big brackets on the last line equal to zero (and noting that w, $V(x)$ and k are all real) implies

$$\begin{cases} V(x) = -w'(x) \\ w''(x) = w(x)V(x) . \end{cases}$$

Substituting the first of these into the second,

$$0 = w''(x) + w(x)w'(x) = \frac{d}{dx}\left(w'(x) + \frac{1}{2}w^2(x)\right) .$$

Integrating once and setting the constant of integration equal to $2\mu^2$ gives us the claimed result.
Substituting $w(x) = 2f'(x)/f(x)$ and cancelling some terms,

$$f''(x) = \mu^2 f(x)$$

and hence $f(x) = Ae^{\mu x} + Be^{-\mu x}$ for some A and B, and

$$w(x) = 2\,\frac{f'(x)}{f(x)} = 2\mu\,\frac{Ae^{\mu x} - Be^{-\mu x}}{Ae^{\mu x} + Be^{-\mu x}} .$$

Since (A, B) and $(\lambda A, \lambda B)$ give the same $w(x)$, we can take $AB = 1$ without loss of generality, and set $A = e^{-\mu x_0}$, $B = e^{\mu x_0}$ for some x_0. Hence

$$w(x) = 2\mu \tanh(\mu(x - x_0)) , \qquad V(x) = -w'(x) = -2\mu^2 \operatorname{sech}^2(\mu(x - x_0)) .$$

As seen earlier in the course (so it won't be repeated here), substituting $u = -V$ into the KdV equation leads to a solution provided that $x_0(t) = x_0(0) + 4\mu^2 t$.

56. Using the results of question 55, show that $V(x) = -2\mu^2 \operatorname{sech}^2(\mu(x-x_0))$ is an example of a reflectionless potential, for which $R(k) = 0$. By adjusting the normalisation of the wavefunction $\psi(x)$ correctly, find out what the transmission coefficient $T(k)$ is for this potential. Verify that $|T(k)|^2 = 1$, consistent with the idea that for such a potential an incident particle must certainly be transmitted.

Solution Substituting $w = 2\mu \tanh(\mu(x - x_0))$ into the given equation we have

$$\psi(x) = 2e^{ikx}(k + i\mu \tanh(\mu(x - x_0))) \sim \begin{cases} 2e^{ikx}(k - i\mu) & x \to -\infty \\ 2e^{ikx}(k + i\mu) & x \to +\infty \,. \end{cases}$$

Dividing through by $2(k - i\mu)$ gives us the correctly-normalised scattering solution:

$$\psi_{\text{scattering}}(x) = e^{ikx}\, \frac{k + i\mu \tanh(\mu(x - x_0))}{k - i\mu} \sim \begin{cases} e^{ikx} & x \to -\infty \\ \frac{k+i\mu}{k-i\mu}\, e^{ikx} & x \to +\infty \end{cases}$$

from which we can read off that $R(k) = 0$ (so the potential is indeed reflectionless) and

$$T(k) = \frac{k + i\mu}{k - i\mu} \,.$$

Furthermore

$$|T(k)|^2 = \frac{|k + i\mu|^2}{|k - i\mu|^2} = \frac{k^2 + \mu^2}{k^2 + \mu^2} = 1$$

as expected.

57. Show by induction or otherwise that the general solution to the differential equation

$$\psi_n''(x) = (-k^2 - n(n+1) \operatorname{sech}^2 x)\psi_n(x) \qquad (n = 0, 1, 2, \dots)$$

is given by $\psi_n(x) = \mathcal{O}_n \mathcal{O}_{n-1} \dots \mathcal{O}_1 \psi_0(x)$, where

$$\psi_0(x) = A(k)e^{ikx} + B(k)e^{-ikx} \,,$$

$A(k)$ and $B(k)$ are constants (with respect to x), and $\mathcal{O}_l$ is the differential operator

$$\mathcal{O}_l = \frac{d}{dx} - l \tanh x \,.$$

Find the asymptotic behaviour of this solution as $x \to \pm\infty$ and hence find the eigenvalues k^2 for the bound states of the potential $V(x) = -n(n+1) \operatorname{sech}^2 x$.

Solution This one is left for you to figure out! (But feel free to ask me for hints.)

58. Let $D = d/dx$ and let $g(x)$ be a general function of x.

   (a) Show that, as differential operators,

$$Dg = g_x + gD \,, \qquad D^2 g = g_{xx} + 2g_x D + gD^2 \,.$$

(b) Show more generally that

$$D^n g = \sum_{m=0}^{n} \binom{n}{m} \frac{d^m g}{dx^m} D^{n-m} \,.$$

[**Hint**: to show that two differential operators are equal, you just have to show that they have the same effect on any function $f(x)$. For part (b), either try induction or think about the formula for the differentiation of a product.]

Solution

(a) g as an operator sends $f(x)$ to $g(x)f(x)$; D sends $f(x)$ to $\frac{d}{dx}f(x)$. Dg means 'do g then do D on the result', so $Dg\,f = \frac{d}{dx}(gf) = g_x f + g f_x = (g_x + gD)f$. Hence on any function f, the action of Dg is the same as that of $g_x + gD$, which implies

$$Dg = g_x + gD \,.$$

Likewise

$$D^2 g\, f = \frac{d^2}{dx^2}(gf) = \frac{d}{dx}(g_x f + g f_x) = g_{xx} f + 2 g_x f_x + g f_{xx}$$

which is the same as $(g_{xx} + 2g_x D + gD^2)f$, from which the desired identity follows.

(b) The relevant formula for differentiating a product is

$$\frac{d^n}{dx^n}(gf) = \sum_{m=0}^{n} \binom{n}{m} g^{(m)} f^{(n-m)} \,.$$

59. Let $D = \partial/\partial x$, and

$$L(u) = D^2 + u(x,t) \,, \qquad M(u) = -(4D^3 + 6uD + 3u_x) \,.$$

Check that

$$L(u)_t + [L(u), M(u)] = u_t + 6uu_x + u_{xxx} \,.$$

Solution This goes as in the lecture notes.

60. Let $L(u) = D^2 + u(x,t)$ and $M(u) = \alpha D$ for some constant α.

(a) Check that

$$L(u)_t = [M(u), L(u)] \quad \Longleftrightarrow \quad u_t = \alpha u_x \,.$$

(b) Let $\psi(x,0)$ be an eigenfunction of $L(u)$ at $t = 0$ with eigenvalue λ, so that

$$(D^2 + u(x,0))\psi(x,0) = \lambda \psi(x,0) \,.$$

If $u(x,t)$ evolves according to the equation of part 1, find an eigenfunction $\psi(x,t)$ for each later time t, with the same eigenvalue λ, so that

$$(D^2 + u(x,t))\psi(x,t) = \lambda \psi(x,t) \,.$$

Verify that $\psi(x,t)$ can be arranged to satisfy $\psi_t = M(u)\psi$. (You can assume that the eigenfunction is non-degenerate, namely that there is a single eigenfunction with that eigenvalue. This is the case both for bound state solutions and for scattering solutions.)

Solution

(a) We have $L(u)_t = u_t$, and

$$[M(u), L(u)] = \alpha[D, D^2 + u] = \alpha[D, u] = \alpha u_x\,.$$

Hence $L(u)_t = [M(u), L(u)] \;\Leftrightarrow\; u_t = \alpha u_x$ as required.

(b) If $u_t = \alpha u_x$ then $u(x,t) = f(x + \alpha t)$; matching to the initial condition at $t = 0$, $u(x,t) = u(x + \alpha t, 0)$. Now suppose that

$$(D^2 + u(x,0))\psi(x,0) = \lambda\psi(x,0)\,.$$

Replacing x by $x + \alpha t$ throughout,

$$(D^2 + u(x + \alpha t, 0))\psi(x + \alpha t, 0) = \lambda\psi(x + \alpha t, 0)$$

but since $u(x,t) = u(x + \alpha t, 0)$ this is the same as

$$(D^2 + u(x,t))\psi(x + \alpha t, 0) = \lambda\psi(x + \alpha t, 0)$$

and hence $(D^2 + u(x,t))\psi(x,t) = \lambda\psi(x,t)$ is solved by setting $\psi(x,t) = \psi(x + \alpha t, 0)$. For this solution we have

$$\psi(x,t)_t = \frac{\partial}{\partial t}\psi(x{+}\alpha t, 0) = \alpha\frac{\partial}{\partial x}\psi(x{+}\alpha t, 0) = \alpha\frac{\partial}{\partial x}\psi(x,t) = \alpha D\psi(x,t) = M(u)\psi(x,t)$$

as required.

61. (a) Show that the differential operator $D = \partial/\partial x$ is anti-symmetric with respect to the inner product

$$\langle\psi_1, \psi_2\rangle := \int_{-\infty}^{+\infty} dx\; \psi_1(x)^* \psi_2(x)$$

on the space $L^2(\mathbb{R})$ of square integrable functions, that is $\langle\psi_1, D\psi_2\rangle = -\langle D\psi_1, \psi_2\rangle$ for all $\psi_1,\ \psi_2 \in L^2(\mathbb{R})$.

(b) Show that $L(u) = D^2 + u(x,t)$ is self-adjoint, given that u is real.

(c) Given a Lax pair $L(u)$, $M(u)$, show that the symmetric part of $M(u)$ commutes with $L(u)$ and therefore drops out of the Lax equation $L(u)_t = [M(u), L(u)]$.

(d) Now assume that $M(u)$ is anti-symmetric. Show that $\langle\psi_1, \psi_2\rangle$ is independent of time t if $\psi_i(x;t)$ evolves according to the equation $(\psi_i)_t = M(u)\psi_i$.

Solution

(a) We have

$$\langle\psi_1, D\psi_2\rangle = \int_{-\infty}^{+\infty} dx\, \psi_1(x)^* \tfrac{\partial}{\partial x}\psi_2(x) = -\int_{-\infty}^{+\infty} dx\, (\tfrac{\partial}{\partial x}\psi_1(x)^*)\psi_2(x) = -\langle D\psi_1, \psi_2\rangle$$

integrating by parts for the middle equality and using the fact that ψ_1 and ψ_2 tend to zero at $\pm\infty$, so there's no boundary term.

(b) This follows in much the same way as part (a), integrating by parts twice.

(c) Given that L and M form a Lax pair, we know that $[L, M]$ is multiplicative (and real) and so $[L, M] = [L, M]^\dagger$, and also $L = L^\dagger$. Hence

$$\begin{aligned} 0 &= [L, M] - [L, M]^\dagger \\ &= LM - ML - (LM - ML)^\dagger \\ &= LM - ML - (M^\dagger L^\dagger - L^\dagger M^\dagger) \\ &= LM - ML - (M^\dagger L - LM^\dagger) \\ &= L(M + M^\dagger) - (M + M^\dagger)L = [L, (M + M^\dagger)] \end{aligned}$$

Since the symmetric part of M is $\frac{1}{2}(M + M^\dagger)$ (and the commutator is linear) the result follows.

(d) We have

$$\begin{aligned} \frac{\partial}{\partial t}\langle\psi_1, \psi_2\rangle &= \langle\frac{\partial}{\partial t}\psi_1, \psi_2\rangle + \langle\psi_1, \frac{\partial}{\partial t}\psi_2\rangle \\ &= \langle M\psi_1, \psi_2\rangle + \langle\psi_1, M\psi_2\rangle \\ &= \langle M\psi_1, \psi_2\rangle - \langle M\psi_1, \psi_2\rangle \qquad \text{(using antisymmetry of } M\text{)} \\ &= 0 \end{aligned}$$

as required.

62. (a) Show that the differential operator of order $2m - 1$

$$M(u) = \sum_{j=1}^{m} \left(\beta_j(x)D^{2j-1} + D^{2j-1}\beta_j(x)\right)$$

is anti-symmetric if the functions $\beta_j(x)$ are real.

(b) If $L(u) = D^2 + u(x, t)$, compute the leading term of $[L(u), M(u)]$ in the form $\gamma(x)D^{2m}$. If $[L(u), M(u)]$ is to be purely multiplicative (forcing $\gamma(x)$ to be zero), deduce that $\beta_m(x)$ must be a constant.

Solution

(a) As in lectures, integration by parts shows that $D^\dagger = -D$, and hence $(D^{2j-1})^\dagger =$

$(-1)^{2j-1}D^{2j-1} = -D^{2j-1}$ if $j \in \mathbb{N}$. Thus

$$\begin{aligned} M(u)^\dagger &= \sum_{j=1}^{m} \left(\beta_j(x)D^{2j-1} + D^{2j-1}\beta_j(x)\right)^\dagger \\ &= \sum_{j=1}^{m} \left((D^{2j-1})^\dagger \beta_j(x)^* + \beta_j(x)^*(D^{2j-1})^\dagger\right) \\ &= -\sum_{j=1}^{m} \left((D^{2j-1})^\dagger \beta_j(x) + \beta_j(x)(D^{2j-1})^\dagger\right) \qquad \text{(since } \beta_j \in \mathbb{R}\text{)} \\ &= -M(u)\,. \end{aligned}$$

(b) For $L(u) = D^2 + u\,, M(u)$ as above, we have

$$\begin{aligned} [L(u), M(u)] &= [D^2 + u, \sum_{j=1}^{m} \left(\beta_j D^{2j-1} + D^{2j-1}\beta_j\right)] \\ &= [D^2, \beta_m D^{2m-1} + D^{2m-1}\beta_m] + \text{(terms involving } D^n \text{ with } n < 2m\text{)} \\ &= [D^2, \beta_m]D^{2m-1} + D^{2m-1}[D^2, \beta_m] + \text{(terms involving } D^n \text{ with } n < 2m\text{)} \end{aligned}$$

where the last step can be checked by writing out the terms. Since $[D^2, \beta_m] = \beta_{m,xx} + 2\beta_{m,x}D$ we deduce

$$\begin{aligned} [L(u), M(u)] &= 2\beta_{m,x}\, D^{2m} + 2D^{2m-1}\beta_{m,x}\, D + \text{(terms involving } D^n \text{ with } n < 2m\text{)} \\ &= 2\beta_{m,x}\, D^{2m} + 2\beta_{m,x}\, D^{2m} + \text{(terms involving } D^n \text{ with } n < 2m\text{)} \\ &= 4\beta_{m,x}\, D^{2m} + \text{(terms involving } D^n \text{ with } n < 2m\text{)}\,. \end{aligned}$$

Now $[L(u), M(u)]$ multiplicative implies in particular that the D^{2m} derivative term must vanish and hence $\beta_{m,x} = 0$, so β_m must be a constant (as a function of x) as required.

63. Consider the $m = 2$ case of the equation from Ex 62 (a). Given the result of that question, you can assume that β_2 is a constant. Fix a normalization by imposing $\beta_2 = 1/2$, and find the most general form of β_1 which allows $[L(u), M(u)]$ to be multiplicative. Show that the Lax equation $L(u)_t + [L(u), M(u)] = 0$ is equivalent to the following alternative version of the KdV equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 2ku_x\,, \qquad (*)$$

where k is an integration constant. Finally, check that the redefined field

$$\tilde{u}(x, t) = u(x + 8kt, -4t)$$

solves the standard KdV equation $\tilde{u}_t + 6\tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0$.

Solution The first part of this is as in example (iii) of section 10.1 in notes (Epiphany term lecture 5). For the last part, using the chain rule for the derivatives we have

$$\tilde{u}_t = 8ku_x - 4u_t\,, \quad \tilde{u}_x = u_x\,, \quad \tilde{u}_{xxx} = u_{xxx}\,,$$

and so

$$\begin{aligned}\tilde{u}_t + 6\tilde{u}\tilde{u}_x + \tilde{u}_{xxx} &= 8ku_x - 4u_t + 6uu_x + u_{xxx} \\ &= 4\Big(-u_t + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x + 2ku\Big) = 0\,,\end{aligned}$$

the term in brackets vanishing by (*).

64. Consider the $m = 3$ case of the equation from problem 62 (a). Given the result of that question, you can assume that $L(u)_t + [L(u), M(u)] = 0$ forces β_3 to be a constant. Complete the calculation to find the most general form of β_2 and β_1 which allow $[L(u), M(u)]$ to be multiplicative. Deduce from a special case of your result that a function $u(x,t)$ evolving according to the fifth-order KdV equation

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0$$

leaves the eigenvalues of $L(u) = D^2 + u$ invariant.

Solution Assuming β_3 is constant as in the question, we can set $\beta_3 = 1/2$ by choice of normalisation. Then

$$M(u) = D^5 + (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)\,.$$

Then $[L(u), M(u)] = [D^2 + u, M(u)]$ and (long calculation – it is handy to use the formula from question 52 to move all the D's to the far right in every term) the terms here are

$$\begin{aligned}[D^2, M(u)] &= [D^2, D^5 + (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)] \\ &= [D^2, (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)] \\ &= 2\beta_{2,x}D^4 + \beta_{2,xx}D^3 + \beta_{2,xxxxx} + 5\beta_{2,xxxx}D + 9\beta_{2,xxx}D^2 + 7\beta_{2,xx}D^3 + 2\beta_{2,x}D^4 \\ &\qquad + \beta_{1,xx}D + 2\beta_{1,x}D^2 + \beta_{1,xxx} + 3\beta_{1,xx}D + 2\beta_{1,x}D^2 \\ &= \beta_{2,xxxxx} + \beta_{1,xxx} + (5\beta_{2,xxxx} + 4\beta_{1,xx})D + (9\beta_{2,xxx} + 4\beta_{1,x})D^2 + 8\beta_{2,xx}D^3 + 4\beta_{2,x}D^4\end{aligned}$$

and

$$\begin{aligned}[u, M(u)] &= [u, D^5 + (\beta_2 D^3 + D^3\beta_2) + (\beta_1 D + D\beta_1)] \\ &= -u_{xxxxx} - 5u_{xxxx}D - 10u_{xxx}D^2 - 10u_{xx}D^3 - 5u_xD^4 \\ &\quad - \beta_2 u_{xxx} - 3\beta_2 u_{xx}D - 3\beta_2 u_x D^2 \\ &\quad - \beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - 3\beta_{2,xx}u_x \\ &\quad - 3(\beta_2 u_{xx} + 2\beta_{2,x}u_x)D - 3\beta_2 u_x D^2 \\ &\quad - 2\beta_1 u_x \\ &= -u_{xxxxx} - 2\beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - (3\beta_{2,xx} + 2\beta_1)u_x \\ &\quad - (5u_{xxxx} + 6\beta_2 u_{xx} + 6\beta_{2,x}u_x)D \\ &\quad - (10u_{xxx} + 6\beta_2 u_x)D^2 \\ &\quad - 10u_{xx}D^3 - 5u_xD^4\,.\end{aligned}$$

Collecting the pieces:

$$\begin{aligned}[L(u), M(u)] = &(4\beta_{2,x} - 5u_x)D^4 + (8\beta_{2,xx} - 10u_{xx})D^3 \\ &+ (9\beta_{2,xxx} + 4\beta_{1,x} - 10u_{xxx} - 6\beta_2 u_x)D^2 \\ &+ (5\beta_{2,xxxx} + 4\beta_{1,xx} - 5u_{xxxx} - 6\beta_2 u_{xx} - 6\beta_{2,x}u_x)D \\ &+ \beta_{2,xxxxx} + \beta_{1,xxx} - u_{xxxxx} - 2\beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - (3\beta_{2,xx} + 2\beta_1)u_x\,.\end{aligned}$$

Next we must set the coefficients of the derivative terms to zero.

D^4: $(4\beta_2 - 5u)_x = 0 \;\Rightarrow\; \beta_2 = \frac{5}{4}(u + k)$ where k is a constant with respect to x.

D^3: now automatic.

D^2: $9\beta_{2,xxx} + 4\beta_{1,x} - 10u_{xxx} - 6\beta_2 u_x = 0$
$\Rightarrow\; \frac{45}{4}u_{xxx} + 4\beta_{1,x} - 10u_{xxx} - \frac{15}{2}(u + k)u_x = \frac{5}{4}u_{xxx} + 4\beta_{1,x} - \frac{15}{2}(u + k)u_x = 0\,,$
$\Rightarrow\; (\frac{5}{4}u_{xx} + 4\beta_1 - \frac{15}{4}u^2 - \frac{15}{2}ku)_x = 0$ and hence $\beta_1 = -\frac{5}{16}\,(u_{xx} - 3u^2 - 6ku + h)$
where h is another constant.

D^1: $(5\beta_{2,xxxx} + 4\beta_{1,xx} - 5u_{xxxx} - 6\beta_2 u_{xx} - 6\beta_{2,x}u_x) = 0.$
It can be checked (bonus exercise!) that this is now automatic.

Finally(!) the D^0 term gives us the general form of $[L(u), M(u)]$, given that it has to be multiplicative:

$$\begin{aligned}[L(u), M(u)] &= \beta_{2,xxxxx} + \beta_{1,xxx} - u_{xxxxx} - 2\beta_2 u_{xxx} - 3\beta_{2,x}u_{xx} - 3\beta_{2,xx}u_x - 2\beta_1 u_x \\ &= \tfrac{5}{4}u_{xxxxx} - \tfrac{5}{16}(u_{xx} - 3u^2 - 6ku)_{xxx} - u_{xxxxx} \\ &\qquad - \tfrac{5}{2}(u + k)u_{xxx} - \tfrac{15}{4}u_x u_{xx} - \tfrac{15}{4}u_{xx}u_x + \tfrac{5}{8}(u_{xx} - 3u^2 - 6ku + h)u_x \\ &= -\tfrac{1}{16}u_{xxxxx} + \tfrac{15}{8}(3u_x u_{xx} + uu_{xxx}) + \tfrac{15}{8}ku_{xxx} \\ &\qquad - \tfrac{5}{2}(u + k)u_{xxx} - \tfrac{15}{4}u_x u_{xx} - \tfrac{15}{4}u_{xx}u_x + \tfrac{5}{8}(u_{xx} - 3u^2 - 6ku + h)u_x \\ &= -\tfrac{1}{16}u_{xxxxx} - \tfrac{5}{4}u_x u_{xx} - \tfrac{5}{8}(u + k)u_{xxx} - \tfrac{15}{8}u^2 u_x - \tfrac{15}{4}kuu_x + \tfrac{5}{8}hu_x\,.\end{aligned}$$

Rescaling $M(u) \to -16M(u)$ and setting $k = h = 0$, we have, applying the general theorem about Lax pairs, that if

$$0 = L(u)_t + [L(u), M(u)] = u_t + u_{xxxxx} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2 u_x$$

then the spectrum of $L(u)$ is independent of t, as required.

65. Let $\psi(x, y)$ be a two-dimensional column vector of smooth functions of x and y, and

$$A = \begin{pmatrix} f & 0 \\ -h & -f \end{pmatrix}\,, \qquad B = \begin{pmatrix} g & h \\ 0 & -g \end{pmatrix}$$

where f, g, h are smooth functions of x and y. Find the consistency conditions for the pair of equations

$$\frac{\partial\psi}{\partial x} + A\psi = 0\,, \qquad \frac{\partial\psi}{\partial y} + B\psi = 0\,.$$

Eliminate f and g from this consistency condition, to leave an equation for h only.

Solution Denote partial derivatives by subscripts. Cross-differentiating the two equations above we find

$$0 = \psi_{yx} + A_y\psi + A\psi_y$$
$$0 = \psi_{xy} + B_x\psi + B\psi_x \ .$$

Using the given pair of equation, we can rewrite this as

$$0 = \psi_{yx} + A_y\psi - AB\psi$$
$$0 = \psi_{xy} + B_x\psi - BA\psi \ .$$

Subtracting and using $\psi_{xy} = \psi_{yx}$ we obtain

$$(A_y - B_x - [A, B])\psi = 0 \ .$$

Hence consistency of the pair of equations requires

$$A_y - B_x - [A, B] = 0 \ .$$

Calculating the partial derivatives and the commutator, we find that this consistency condition reads

$$\begin{pmatrix} f_y - g_x - h^2 & -h_x - 2fh \\ -h_y + 2gh & -f_y + g_x + h^2 \end{pmatrix} = 0 \ .$$

Solving the off-diagonal equations,

$$h_x + 2fh = 0 \quad \Longrightarrow \quad f = -\frac{1}{2}(\log h)_x$$
$$-h_y + 2gh = 0 \quad \Longrightarrow \quad g = +\frac{1}{2}(\log h)_y \ .$$

The diagonal equations coincide. Using the expressions for f and g that we just derived, the equation is

$$0 = f_y - g_x - h^2 = -\frac{1}{2}(\log h)_{yx} - \frac{1}{2}(\log h)_{xy} - h^2 \ ,$$

hence the PDE for h is

$$(\log h)_{xy} + h^2 = 0 \ .$$

66. Two differential operators are given as

$$L = \partial_y - \partial_x^2 - u$$
$$M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x + w$$

where $u = u(t, x, y)$ and $w = w(t, x, y)$. Show that if $[M, L]\psi = 0$ for any smooth ψ, then

$$w_y = -u_t - u_{xxx} - 6uu_x$$
$$w_x = 3u_y \ .$$

Show that there exists a solution for u, w of the form

$$u = \frac{A}{\cosh^2(z)}\ , \quad w = \frac{B}{\cosh^2(z)}$$

where $z = \alpha t + \beta x + \gamma y$ and $\alpha, \beta, \gamma, A, B$ are constants satisfying $A = 2\beta^2$, $B = 6\beta\gamma$ and $\alpha\beta + 4\beta^4 + 3\gamma^2 = 0$.

Solution This is left as an exercise.

67. The Lax Pair L, M is defined by the expressions

$$\begin{aligned} L &= (1-c)\,\partial_x + au + bv \\ M &= \partial_x^2 + \frac{1}{2}(u^2+v^2) - av_x + bu_x \end{aligned}$$

where where u, v are real valued functions and a, b, c are constant unit quaternions, that you may take to be defined by their (non-commutative) multiplication rules:

$$a^2 = b^2 = c^2 = -1\ , \quad ab = c\ , \quad bc = a\ , \quad ca = b\ .$$

Note that a, b, c do not commute (*e.g.* $ab \neq ba$). Find the differential equations satisfied by u and v which are a sufficient condition for the eigenvalue of L to be time-independent.

[**Hint:** Equations involving quaternions are a generalization of complex equations in the sense that one can equate coefficients of $1, a, b, c$.]

Solution This is left as an exercise for those who like unusual number systems.

68. The functional derivative $\delta F/\delta u$ of $F[u]$ is defined by the equation

$$F[u+\delta u] = F[u] + \int_{-\infty}^{+\infty} dx\, \frac{\delta F[u]}{\delta u(x)} \delta u(x) + \mathcal{O}((\delta u)^2)\ ,$$

where the infinitesimal variation $\delta u(x)$ is small everywhere and goes to zero at the boundaries of the integration range (the same applies to its derivatives $\delta u_x,\ \delta u_{xx},\ \ldots$). If

$$F[u] = \int_{-\infty}^{+\infty} dx\ f(u, u_x, u_{xx}, u_{xxx}, \ldots)\ ,$$

show that

$$\frac{\delta F[u]}{\delta u} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} - \frac{\partial^3}{\partial x^3}\frac{\partial f}{\partial u_{xxx}} + \ldots$$

Solution We have

$$\begin{aligned}
F[u+\delta u] &= \int_{-\infty}^{+\infty} dx\, f(u+\delta u, u_x+\delta u_x, u_{xx}+\delta u_{xx}, \dots) \\
&= \int_{-\infty}^{+\infty} dx\, f(u) + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u_x}\delta u_x + \frac{\partial f}{\partial u_{xx}}\delta u_{xx} + \dots + \mathcal{O}((\delta u)^2) \\
&= F[u] + \int_{-\infty}^{+\infty} dx\, \frac{\partial f}{\partial u}\,\delta u - \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial u_x}\Big)\,\delta u + \frac{\partial^2}{\partial x^2}\Big(\frac{\partial f}{\partial u_{xx}}\Big)\,\delta u + \dots + \mathcal{O}((\delta u)^2) \\
&= F[u] + \int_{-\infty}^{+\infty} dx\, \left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial u_x}\Big) + \frac{\partial^2}{\partial x^2}\Big(\frac{\partial f}{\partial u_{xx}}\Big) + \dots\right)\delta u + \mathcal{O}((\delta u)^2)
\end{aligned}$$

where to get from the second line to the third we integrated by parts once for the δu_x term, twice for δu_{xx}, and so on, each time using the fact that the variation and its dervitives go to zero at the boundaries of the integration range. Comparing the last line with the formula in the question gives the desired result.

69. (a) Find a function $f(u, u_x, u_{xx})$ and a functional

$$F[u] = \int_{-\infty}^{+\infty} dx\, f(u, u_x, u_{xx})$$

such that the equation

$$u_t = \frac{\partial}{\partial x}\frac{\delta F}{\delta u}$$

is the same as the fifth-order KdV equation from question 64.

(b) Show that your $F[u]$ is a conserved quantity if u evolves according to the standard third order KdV equation.

(c) Show that $\int_{-\infty}^{+\infty} dx\, u$ is a conserved quantity if u evolves according to the fifth-order KdV equation.

Solution See the handwritten notes for problems class 7.

70. Consider the scattering data

$$S = \{R(k), \quad \{\mu_n, c_n\}_{n=1}^{N}\}$$

for the potential $V(x) = a\delta(x)$ derived in the lectures. For each sign of a:

(a) Calculate

$$F(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, R(k)\, e^{-ik\xi} + \sum_{n=1}^{N} c_n^2\, e^{\mu_n \xi}\,.$$

[**Hint:** close the integration contour of the k integral by adding an infinite arc in the upper or lower half of the complex plane for k, and use Cauchy's residue theorem.]

(b) Solve the Marchenko equation

$$K(x,z)+F(x{+}z)+\int_{-\infty}^{x}dy\,K(x,y)\,F(y{+}z)=0$$

to determine the unknown function $K(x,z)$ for all $z\le x$ (and set $K(x,z)=0$ for $x<z$).

(c) Show that

$$V(x)=2\frac{d}{dx}\lim_{z\to x^-}K(x,z)\ .$$

Solution For $a>0$, see the handwritten notes for problems class 7.

For $a<0$, the main difference is that there is a single bound state. From the scattering data

$$S=\left\{R(k)=\frac{a}{2ik-a},\left\{\mu_1=-\frac{a}{2},c_1=\sqrt{-\frac{a}{2}}\right\}\right\}\ ,$$

we find

$$F(\xi)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{a}{2ik-a}e^{-ik\xi}-\frac{a}{2}e^{-\frac{a}{2}\xi}\ .$$

The integrand has a pole at $k=-ia/2$, which has a positive imaginary part. As in the case $a>0$ which was solved in the problems class, we close the integration contour (originally the real k axis) in the complex plane by adding an infinite arc in the upper or lower half plane, picking the arc in such a way that it doesn't contribute to the integral, thanks to exponential damping.

- $\underline{\xi>0}$: we have

  $$|e^{-ik\xi}|=e^{\xi\ \mathrm{Im}(k)}\ ,$$

  therefore the integrand tends to zero exponentially fast along the lower infinite arc, where $\mathrm{Im}(k)\to-\infty$. Thus we find

  $$F(\xi)=-\frac{a}{2}\oint_{C_-}\frac{dk}{2\pi i}\frac{e^{-ik\xi}}{k+i\frac{a}{2}}-\frac{a}{2}e^{-\frac{a}{2}\xi}\ ,$$

  where C_- is the counterclockwise (and $-C_-$ the clockwise) oriented contour in the complex k plane consisting of the infinite arc in the lower half plane and the real line. Because the integrand is holomorphic (that, is it has no poles) in the region enclosed by C_-, the integral vanishes by Cauchy's residue theorem, and we obtain

  $$F(\xi)=-\frac{a}{2}e^{-\frac{a}{2}\xi}\ .$$

- $\underline{\xi<0}$: we have

  $$|e^{-ik\xi}|=e^{\xi\ \mathrm{Im}(k)}\ ,$$

  therefore the integrand tends to zero exponentially fast along the upper infinite arc, where $\mathrm{Im}(k)\to+\infty$. Adding (at no cost) this arc to the real k line we find that

  $$F(\xi)=\frac{a}{2}\oint_{C_+}\frac{dk}{2\pi i}\frac{e^{-ik\xi}}{k+i\frac{a}{2}}-\frac{a}{2}e^{-\frac{a}{2}\xi}\ ,$$

where C_+ is the counterclockwise contour in the complex k plane consisting of the real line followed by the infinite arc in the upper half plane, until the contour is closed. Now the integration contour contains the single simple pole of the integrand (at $k = -ia/2$), and by Cauchy's residue theorem we find that

$$\begin{aligned} F(\xi) &= \frac{a}{2} \operatorname*{Res}_{k=-i\frac{a}{2}} \frac{e^{-ik\xi}}{k + i\frac{a}{2}} - \frac{a}{2} e^{-\frac{a}{2}\xi} \\ &= \frac{a}{2} e^{-\frac{a}{2}\xi} - \frac{a}{2} e^{-\frac{a}{2}\xi} \ = 0 \ . \end{aligned}$$

We can combine the results for $\xi > 0$ and $\xi < 0$ together in the single formula

$$F(\xi) = -\frac{a}{2} e^{-\frac{a}{2}\xi}\, \Theta(\xi) \ ,$$

where $\Theta(\xi)$ is the Heaviside step function.

We have found exactly the same result for $F(\xi)$ that we found for $a > 0$ in Problems Class 7. The solution of parts (b) and (c) is then identical to the one for $a > 0$. See the handwritten notes for Problems Class 7 for the details.

[**Note:** the careful reader may complain that we didn't determine what happens if $\xi = 0$. We run into the problem that the integral in

$$\begin{aligned} F(0) &= -i \frac{a}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{k + i\frac{a}{2}} - \frac{a}{2} \\ &= -i \frac{a}{4\pi} \int_{-\infty}^{\infty} dk \left[\frac{k}{k^2 + \frac{a^2}{4}} - i \frac{a}{2} \frac{1}{k^2 + \frac{a^2}{4}} \right] - \frac{a}{2} \end{aligned}$$

is not absolutely convergent. If we regularize the imaginary part of the integral using Cauchy's principal value, then this imaginary part vanishes (because we integrate an odd function over a symmetric domain). The real part of the integral is $a/4$, therefore we get $F(0) = -a/4$. This is consistent with taking $\Theta(0) = \frac{1}{2}$, the average of the left-sided and right-sided limits, in the general formula above.]

71. Consider the scattering data

$$S = \{R(k), \quad \{\mu_n, c_n\}_{n=1}^{N}\}$$

for the square barrier/well potential studied in problem 53. For each sign of V_0:

(a) Calculate

$$F(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, R(k)\, e^{-ik\xi} + \sum_{n=1}^{N} c_n^2\, e^{\mu_n \xi} \ .$$

(b) Solve the Marchenko equation

$$K(x,z) + F(x+z) + \int_{-\infty}^{x} dy\, K(x,y)\, F(y+z) = 0$$

to determine the unknown function $K(x,z)$ for all $z \le x$ (and set $K(x,z) = 0$ for $x < z$).

(c) Show that

$$V(x) = 2\frac{d}{dx}\lim_{z\to x^-} K(x,z) \ .$$

Solution This is left for you as an exercise. Feel free to ask me hints.

72. Show that the Poisson bracket $\{,\}$ has the following properties for any three smooth functions f, g and h on phase space:

• antisymmetry:

$$\{f,g\} = -\{g,f\}$$

• bilinearity:

$$\{af+bg,h\} = a\{f,h\}+b\{g,h\}\ , \quad \{f,ag+bh\} = a\{f,g\}+b\{f,h\}\ , \qquad a,b\in\mathbb{R}$$

• Leibniz's rule:

$$\{fg,h\} = \{f,h\}g + f\{g,h\}$$

• The Jacobi identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

73. Let $Q_1(p,q)$ and $Q_2(p,q)$ be two smooth functions on phase space in involution, that is $\{Q_1,Q_2\}=0$. Q_1 and Q_2 generate the evolution of the Hamiltonian system under two different 'time' coordinates t_1 and t_2 respectively, according to the equations

$$\frac{\partial p_i}{\partial t_a} = -\frac{\partial Q_a}{\partial q_i}$$
$$\frac{\partial q_i}{\partial t_a} = +\frac{\partial Q_a}{\partial p_i}$$

where $a=1,2$.

(a) Show that the evolution of a smooth function $X(p,q)$ by an infinitesimal time dt_a generated by Q_a is given by

$$X \mapsto X + \{Q_a,X\}dt_a + \frac{1}{2}\{Q_a,\{Q_a,X\}\}dt_a^2 + \mathcal{O}(dt_a^3)\ ,$$

to second order in the infinitesimal time increment dt_a.

(b) Evolve X first by an infinitesimal time dt_1 using Q_1, and then by an infinitesimal time dt_2 using Q_2, working to second order in dt_1 and dt_2.
[**Note:** you'll need to keep terms proportional to dt_1^2, to dt_2^2 and to $dt_1 dt_2$.]

(c) Repeat the time evolutions in the opposite order: first by dt_2 using Q_2, and then by dt_1 using Q_1.

(d) Show that the results of parts (b) and (c) coincide.
[**Hint:** use the Jacobi identity for the Poisson bracket.]

74. If

$$L(t) = \begin{pmatrix} x(t) & y(t) \\ y(t) & -x(t) \end{pmatrix} ,$$

find an antisymmetric matrix $M(t)$ such that the Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the system of ODE's

$$\begin{cases} \dot{x} = gy \\ \dot{y} = -gx \end{cases}$$

where $g(x, y, t)$ is some function of x, y and t, and dots denote time derivatives. Using only the symmetry properties of L, together with the Lax equation, show that the eigenvalues of L do not depend on t. Deduce the (otherwise fairly obvious) fact that if $x(t)$ and $y(t)$ evolve according to the above system of ODE, then the value of $x(t)^2 + y(t)^2$ remains constant.

75. Consider a classical Hamiltonian system with $n = 3$ coordinates q_i and momenta p_i. A Lax pair of matrices L and M is given by

$$L = \begin{bmatrix} p_1 & b_1 & b_3 \\ b_1 & p_2 & b_2 \\ b_3 & b_2 & p_3 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & b_1 & -b_3 \\ -b_1 & 0 & b_2 \\ b_3 & -b_2 & 0 \end{bmatrix},$$

where $p_i = \dot{q}_i$ and $b_i = \exp[c(q_i - q_{i+1})]$ for some constant c (with $q_{i+3} = q_i$ and $p_{i+3} = p_i$). Use the Lax equation $\dot{L} + [L, M] = 0$ to find the constant c and to obtain equations of motion in the form $\ddot{q}_i = f_i(q)$, for some functions $f_i(q)$ that you should determine.

A Useful integrals

You can freely quote the following formulae, although deriving them may be instructive:

- **Indefinite integrals:** [**Note:** the integration constant is in principle complex]

$$\int \frac{dx}{x\sqrt{1-x}} = -2\text{arcsech}(\sqrt{x}) \tag{1}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\text{arcsech}(x) \tag{2}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\text{arccosech}(x) \tag{3}$$

$$\int \frac{dx}{\sin(x/2)} = 2\log\tan(x/4) \tag{4}$$

$$\int \frac{dx}{\cosh(x)} = 2\arctan(e^x) \tag{5}$$

$$\int \frac{dx}{1-x^2} = \text{arctanh}(x) \tag{6}$$

$$\int dx\,\sqrt{1-x^2} = \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin(x)\right] \tag{7}$$

$$\int \frac{dx}{\cos^2(x)} = \tan(x) \tag{8}$$

$$\int \frac{dx}{\cosh^2(x)} = \tanh(x) \tag{9}$$

- **Definite integrals:**

$$\int_{-\infty}^{+\infty} dx\; e^{-Ax^2} = \sqrt{\frac{\pi}{A}} \qquad (A>0) \tag{10}$$

$$\int_{-\infty}^{+\infty} dx\,\text{sech}^{2n}(x) = \frac{2^{2n-1}((n-1)!)^2}{(2n-1)!} \tag{11}$$

Note: the result of the Gaussian integral (10) does not change if the integration variable x is shifted by a finite imaginary amount c, namely if you replace $x \to x + ic$.