DISPERSIONLESS KdV $\quad u_t + 6uu_x = 0$

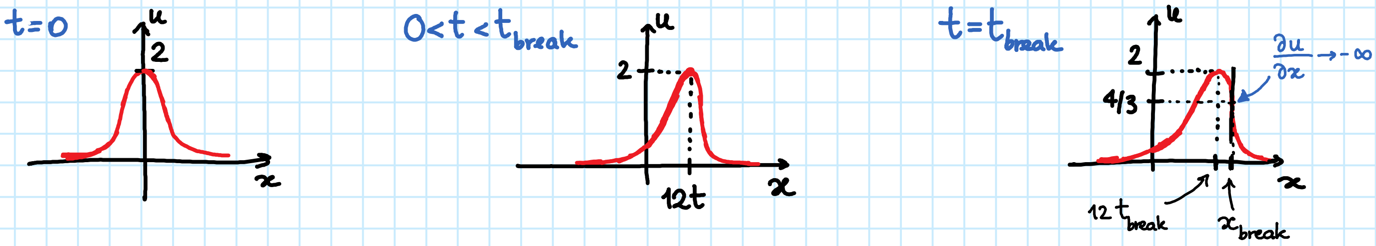
I want to determine at what time the solution with initial condition $u(x,0) = \frac{2}{\cosh^2 x}$ develops a steep front and breaks. This can be done using the method of characteristics.

$$\nabla u = (u_t, u_x) \overset{\text{dKdV}}{=} u_x(-6u, 1) \qquad \perp (1, 6u)$$

$$\Rightarrow \quad u = \text{const. along curve } \frac{dx}{dt} = 6u. \quad \Rightarrow \quad u(x,t) = f(x - 6ut)$$

For the given initial condition, $\quad u(x,t) = \dfrac{2}{\cosh^2(x - 6ut)}$.

$$\Rightarrow \quad \boxed{x = 6ut \pm \operatorname{arccosh}\sqrt{\frac{2}{u}}} \qquad x \in \mathbb{R} \Rightarrow u \in (0, 2].$$



The solution will break at the special value of t for which $\frac{\partial u}{\partial x} \to \pm\infty$ at some x.

Therefore we look for solutions of $\frac{\partial x}{\partial u} = 0$:

$$0 = 6t \mp \frac{1}{\sqrt{2}\sqrt{2-u}\,u} \quad \Rightarrow \quad t = \pm \frac{1}{6}\frac{1}{\sqrt{2}\sqrt{2-u}\,u} \qquad \left(\begin{array}{l}\text{and we can focus}\\ \text{on } t>0\end{array}\right)$$

The breaking time is the smallest value of t for which $\left|\frac{\partial u}{\partial x}\right| \to \infty$, therefore it is the minimum of $\frac{1}{6}\frac{1}{\sqrt{2}\sqrt{2-u}\,u}$ for $u \in (0, 2]$.

$$0 = \frac{d}{du}\left[\frac{1}{\sqrt{2-u}\,u}\right] = \frac{3u-4}{2u^2(2-u)^{3/2}} \quad \rightsquigarrow \quad u = u_{break} \equiv \frac{4}{3}$$

$$\rightsquigarrow \qquad t_{break} = \frac{\sqrt{3}}{16} \approx 0.108253\ldots$$

$$x_{break} = \frac{\sqrt{3}}{2} + \operatorname{arccosh}\sqrt{\frac{3}{2}} \simeq 1.5245$$