Assignment 1

Due date: Wednesday, 20 October (8pm)

Ex 2

This exercise explores the scale invariance of the KdV equation.

1. Show that if $u(x,t) = g(x,t)$ solves the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \ ,$$

so does $u(x,t) = Ag(Bx, Ct)$, provided that the constants B and C are related to A in a way which you should determine. [15 marks]

SOLUTION:

Let $u(x,t) = Ag(X,T)$, where $X = Bx$ and $T = Ct$. By the chain rule

$$u_t(x,t) = ACg_T(X,T) \ , \quad u_x(x,t) = ABg_X(X,T) \ , \quad u_{xxx}(x,t) = AB^3 g_{XXX}(X,T) \ ,$$

so the left-hand side of the KdV equation for $u(x,t) = Ag(Bx, Ct)$ becomes

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= ACg_T(X,T) + 6A^2Bg(X,T)g_X(X,T) + AB^3 g_{XXX}(X,T) \\ &= AC\left[g_T(X,T) + \frac{AB}{C} \cdot 6g(X,T)g_X(X,T) + \frac{B^3}{C} \cdot g_{XXX}(X,T)\right] \ . \end{aligned}$$

If $AB/C = B^3/C = 1$, this becomes

$$= AC\left[g_T(X,T) + 6g(X,T)g_X(X,T) + g_{XXX}(X,T)\right] \ ,$$

which vanishes because $g(X,T)$ solves the KdV equation $g_T + 6gg_X + g_{XXX} = 0$ in its variables X and T by assumption. The two algebraic equations among the parameters are solved by $A = B^2$ and $C = B^3$. So

$$u(x,t) = g(x,t) \quad \text{solves KdV} \quad \Longrightarrow \quad u(x,t) = B^2 g(Bx, B^3 t) \quad \text{solves KdV} \ \ \forall B \in \mathbb{R} \ .$$

2. Apply this transformation to the basic KdV solution

$$u(x,t) = \frac{2}{\cosh^2(x-4t)}$$

to construct a one-parameter family of one-soliton solutions of the KdV equation. [10 marks]

SOLUTION:

Here $g(x,t) = 2\text{sech}^2(x-4t)$. By the above logic,

$$u(x,t) = 2B^2 \cdot \text{sech}^2\left(B(x-4B^2t)\right)$$

is a family of solutions of the KdV equation labelled by a single real parameter B, which we can take to be positive wlog since sech^2 is an even function.

3. Find a formula relating the velocities to the heights for solitons in this one-parameter family. How does the spatial width of a soliton in this family change if its velocity is rescaled by a factor of 4? [10 marks]

SOLUTION:

The height is the maximum of u, and the velocity v is read off from the dependence on x, t through the single linear combination $x - vt$. We find that height $= 2B^2$ and velocity $= 4B^2$, so velocity $= 2 \times$ height.
The width is a measure of how much the lump is concentrated in space. Since the dependence on the spatial coordinate x is only through Bx, we deduce that width $\sim 1/B$.[1] The precise proportionality factor depends on the precise definition of width that you might choose, but regardless of that choice

$$\text{velocity} \mapsto 4 \times \text{velocity} \equiv B^2 \times \text{velocity} \quad \Longrightarrow \quad \text{width} \mapsto \frac{1}{B} \times \text{width} = \frac{1}{2} \times \text{width}\ .$$

Ex 4

Consider a pair of solitons with velocities m and n in the ball and box model, with $m > n$ and the faster soliton to the left of the slower one, with separation $l \geq n$ (*i.e.* there are l empty boxes between the two solitons). Evolve various such initial conditions forward in time using the ball and box rule, for different values of m, n and l. Prove that the system always evolves into an oppositely-ordered pair of the same two solitons, and find a general formula for the phase shifts of the solitons in terms of m and n. [45 marks]

SOLUTION:

[1] If you thought that width $\sim B$ rather than $1/B$, pick your favourite localised function $f(x)$, plot $f(x)$ and $f(2x)$ and compare: is the width of $f(2x)$ double or half the width of $f(x)$?

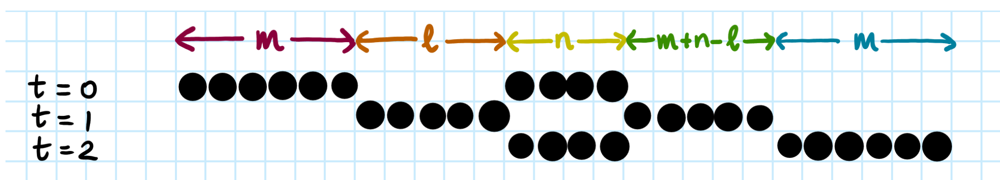
Rather than discussing examples, I will give a proof and calculate the phase shifts in full generality, but you will get credit for providing examples as long as they are correct.
The velocity m soliton has length m and (by definition) moves by m boxes in one unit of time when it is far away from other solitons. Likewise, the length n soliton moves by n boxes in one unit of time when it is far away from other solitons. Therefore, when the faster length m soliton is far enough behind the slower length n soliton, the separation decreases as follows in one unit of time (the subscript denotes the value of the discrete time coordinate):

$$l_t = l \quad \mapsto \quad l_{t+1} = l - m + n \ .$$

We can iterate this process until the separation l reaches the range

$$n \leq l < m \ ,$$

at a time that I will label as $t = 0$ in the following (this can be achieved by a shift of the time coordinate). Let's now evolve the system forward from $t = 0$, using the above inequality and taking into account that boxes which are full at time t are empty at time $t + 1$:



At $t = 2$ the slower length n soliton is more than n boxes behind the faster length m soliton, so the collision is over and we don't need to evolve the system any further to calculate the phase shifts. At $t = 2$ the faster length m soliton has moved by $m + l + n + (m + n - l) = 2(m + n)$ boxes compared to where it was at $t = 0$. In the absence of the slower soliton, it would have moved by $2m$ boxes. The slower length n soliton has not moved at $t = 2$ compared to where it was at $t = 0$. In the absence of the faster soliton, it would have moved by $2n$ boxes. The differences between the positions of the solitons after the collision and the positions the solitons would have had before the collision are the phase shifts

$$\begin{aligned}(\text{phase shift})_{\text{faster}} &= 2(m + n) - 2m = 2n \ , \\ (\text{phase shift})_{\text{slower}} &= 0 - 2n = -2n \ .\end{aligned}$$

Ex 11

Find the dispersion relation and the phase and group velocities for:

(a) $\quad u_t + u_x + \alpha u_{xxx} = 0$ [10 marks]

SOLUTION:

Sub in a plane wave $u(x,t) = e^{i(kx-\omega t)}$ to get the algebraic equation $-i\omega + ik - i\alpha k^3 = 0$. So the dispersion relation is

$$\omega = \omega(k) = k - \alpha k^3 = k(1-\alpha k^2) \ ,$$

and the phase and group velocity are

$$\begin{aligned} \text{Phase velocity :} \qquad c(k) \ &= \frac{\omega(k)}{k} = 1 - \alpha k^2 \\ \text{Group velocity :} \qquad c_g(k) &= \omega'(k) = 1 - 3\alpha k^2 \ . \end{aligned}$$

(b) $$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}$$ [10 marks]

SOLUTION:

Sub in a plane wave $u(x,t) = e^{i(kx-\omega t)}$ to get the algebraic equation $-\omega^2 + \alpha^2 k^2 = \beta^2 k^2 \omega^2$. So the dispersion relation is[2]

$$\omega = \omega(k) = \pm \frac{\alpha k}{(1+\beta^2 k^2)^{1/2}} \ ,$$

and the phase and group velocity are

$$\begin{aligned} \text{Phase velocity :} \qquad c(k) \ &= \frac{\omega(k)}{k} = \pm \frac{\alpha}{(1+\beta^2 k^2)^{1/2}} \\ \text{Group velocity :} \qquad c_g(k) &= \omega'(k) = \pm \alpha \left[\frac{1}{(1+\beta^2 k^2)^{1/2}} - \frac{k}{2} \frac{2\beta^2 k}{(1+\beta^2 k^2)^{3/2}} \right] \\ &= \pm \frac{\alpha}{(1+\beta^2 k^2)^{3/2}} \ . \end{aligned}$$

[2] We might restrict to the positive solution for the dispersion relation as in the lecture notes, since the negative solution is obtained by taking the complex conjugate of the plane wave and reversing the sign of k. I won't do it in this solution, but your marks will not be affected by the sign you picked whether you wrote $+$ or $\pm$.