Assignment 2

Due date: Tuesday, 1 November (8pm)

Ex 13

Find (if they exist) real non-singular travelling wave solutions of the following equations, satisfying the given boundary conditions:

1. Modified KdV (mKdV) equation:

$$\begin{aligned} &u_t + 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\ . \end{aligned} \tag{2.1}$$

[50 marks]

SOLUTION:

Substitute in the travelling wave solution $u(x,t) = f(x - vt) \equiv f(\xi)$, to get the ODE

$$-vf' + 6f^2 f' + f''' = 0$$

and the boundary conditions $f, f', f'' \to 0$ as $\xi \to \pm\infty$. Now integrate the equation twice (the second time with an integrating factor f'):

$$-vf + 2f^3 + f'' = A$$
$$-\frac{v}{2}f^2 + \frac{1}{4}f^4 + \frac{1}{2}(f')^2 = Af + B\ ,$$

where A, B are integration constants. The boundary conditions determine $A = B = 0$, so we find

$$\frac{1}{2}(f')^2 + \underbrace{\frac{1}{2}f^2(f^2 - v)}_{=\hat{V}(f)} = \underbrace{0}_{=\hat{E}}$$

A real solution requires $\hat{V}(f) \leq \hat{E} \equiv 0$. A value f_0 of f such that $\hat{V}(f_0) = \hat{E} \equiv 0$ is an "equilibrium point" if $\frac{d\hat{V}}{df}(f_0) = 0$ and a "turning point" if $\frac{d\hat{V}}{df}(f_0) \neq 0$. The former are allowed boundary values for f as $\xi \to \pm\infty$, indeed all derivatives of $f(\xi)$ vanish. The latter are reached for an instant at a finite value ξ_0 of ξ, after which $f'(\xi)$ changes sign.

For the problem at hand we need to consider two cases:

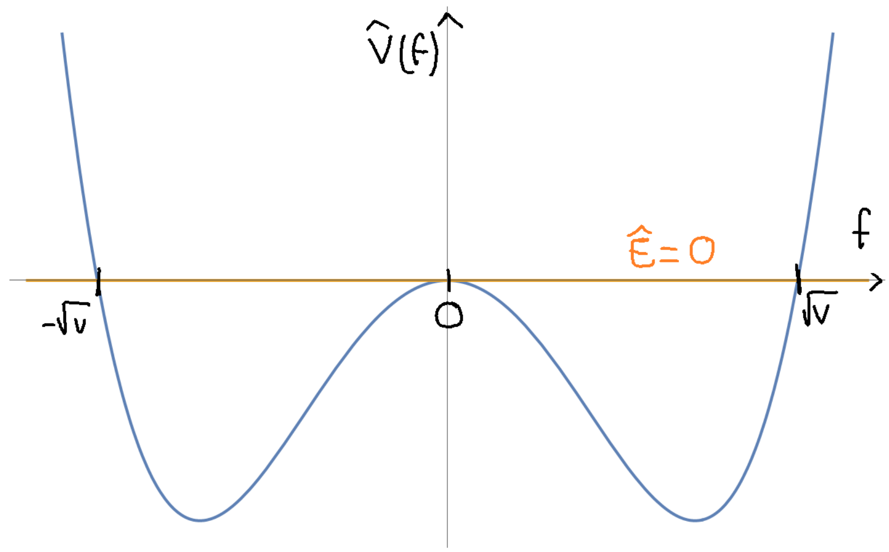

Figure 1: Graph of $\hat{V}(f) = \frac{1}{2}f^2(f^2 - v)$ (blue) and $\hat{E} = 0$ (orange).

(a) $\boxed{v \leq 0}$:
in this case $\hat{V}(f) \geq 0 = \hat{E}$ for all $f \in \mathbb{R}$, and $\hat{V}(f) = 0$ iff $f = 0$. So the only real solution is the constant solution $f(\xi) = 0$.

(b) $\boxed{v > 0}$:
in this case $\hat{V}(f) \leq 0 = \hat{E}$ for all $f \in \mathbb{R}$, and $\hat{V}(f) = 0$ if $f \in [-\sqrt{v}, +\sqrt{v}]$. $f = 0$ is an "equilibrium point", whereas $f = \pm\sqrt{v}$ are turning points. See figure 1. There are three real solutions, all of which are non-singular, corresponding to $f'(\xi)$ being vanishing, positive or negative in an open neighbourhood of $\xi = -\infty$:

- the constant solution $f(\xi) = 0$;
- a positive solution, in which f grows from 0^+ as $\xi \to -\infty$ to a maximum $+\sqrt{v}$ at a finite value $\xi = x_0$, and then decreases to 0^+ as $\xi \to +\infty$;
- a negative solution, in which f decreases from 0^- as $\xi \to -\infty$ to a minimum $-\sqrt{v}$ at a finite value $\xi = x_0$, and then increases to 0^- as $\xi \to +\infty$.

Let's focus on the positive solution which obeys $0 < f(\xi) \leq \sqrt{v}$, since the negative solution can be found by changing sign to f. We need to solve

$$f' = \pm\sqrt{2(\hat{E} - \hat{V}(f))} = \pm f\sqrt{v - f^2} \ ,$$

where the positive root is taken as f increases from 0^+ to $\sqrt{v}$ and the negative root is taken as f decreases from $\sqrt{v}$ to 0^+. Separating variables we find:

$$\int \frac{df}{f\sqrt{v - f^2}} = \pm \int d\xi = \pm(\xi - x_0) \ .$$

Using the indefinite integral[1]

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\mathrm{arcsech}(x) \ ,$$

we find

$$-\frac{1}{\sqrt{v}}\mathrm{arcsech}\left(\frac{f}{\sqrt{v}}\right) = \pm(\xi - x_0) \ ,$$

which we can invert to

$$f(\xi) = \sqrt{v}\ \mathrm{sech}\left[\mp\sqrt{v}(\xi - x_0)\right] = \sqrt{v}\ \mathrm{sech}\left[\sqrt{v}(\xi - x_0)\right] \ .$$

So the two non-trivial solutions are

$$u(x,t) = \pm\sqrt{v}\ \mathrm{sech}\left[\sqrt{v}(x - x_0 - vt)\right] \ .$$

3. "ϕ^4 theory":

$$\begin{aligned} &u_{tt} - u_{xx} + 2u(u^2-1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to -1 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to +1 \ \text{ as } x \to +\infty \ . \end{aligned} \tag{2.2}$$

[50 marks]

SOLUTION:

Substitute in the travelling wave solution $u(x,t) = f(x - vt) \equiv f(\xi)$, to get the ODE

$$(1-v^2)f'' = 2f(f^2-1)$$

or equivalently

$$f'' = 2\gamma^2 f(f^2-1) \ ,$$

where $\gamma = (1-v^2)^{-1/2}$ is the Lorentz factor. Integrate this equation once with an integrating factor f' to get

$$\frac{1}{2}(f')^2 = \gamma^2\left(\frac{1}{2}f^4 - f^2\right) + A \ .$$

Now impose the boundary conditions ($e.g. f \to 1,\ f' \to 0$ as $\xi \to +\infty$) to determine the integration constant A:

$$0 = \gamma^2\left(\frac{1}{2} - 1\right) + A \quad \Longrightarrow \quad A = \frac{\gamma^2}{2} \ .$$

[1]Subtlety: $\mathrm{arcsech}(-x) - \mathrm{arcsech}(x)$ is piecewise constant (and purely imaginary), so you can also take $\mathrm{arcsech}(-x)$ as the result of that integral. That's what you would use if you were after the negative solution. If you are interested in real indefinite integrals, it's perhaps better to write the result of the indefinite integral as $-\mathrm{arctanh}(\sqrt{1-x^2})$, which is manifestly real and even for $x \in [-1,0) \cup (0,+1]$.

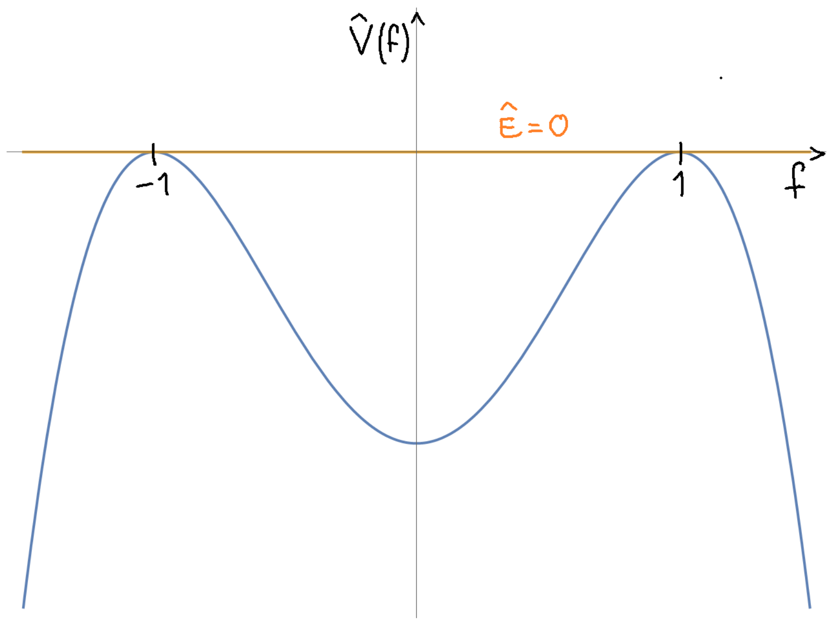

Figure 2: Graph of $\hat{V}(f) = -\frac{1}{2}\gamma^2(f^2-1)^2$ (blue) and $\hat{E}=0$ (orange).

Substituting back in the previous equation we get

$$\frac{1}{2}(f')^2 \underbrace{-\frac{1}{2}\gamma^2(f^2-1)^2}_{=\hat{V}(f)} = \underbrace{0}_{=\hat{E}} ,$$

which leads to

$$f' = \pm\gamma(f^2-1) .$$

From the graph of $\hat{V}(f)$ in fig. 2, we see that $\hat{V}(f) \le \hat{E} = 0$ for all $f \in \mathbb{R}$, and that there are two "equilibrium points" for f, namely $f = \pm 1$, which are allowed boundary values of f. There exist four non-singular travelling wave solutions which interpolate between those values of f: the two constant solutions $f(\xi) = \pm 1$, a solution which interpolates from $f = -1$ as $\xi \to -\infty$ to $f = +1$ as $\xi \to +\infty$, and a solution which interpolates between $f = +1$ as $\xi \to -\infty$ to $f = -1$ as $\xi \to +\infty$.
The third of these solutions obeys the given boundary conditions. Let's find it. It has $-1 < f < 1$ and $f' > 0$ for all $\xi \in \mathbb{R}$, so it obeys the ODE

$$f' = \gamma(1-f^2) ,$$

which is solved by separation of variables:

$$\int \frac{df}{1-f^2} = \gamma \int d\xi$$
$$\operatorname{arctanh}(f) = \gamma(\xi - x_0)$$

leading to the travelling wave solution

$$u(x,t) = f(x - vt) = \tanh\left[\gamma(x - x_0 - vt)\right] \ .$$

It's easy to check that this solution indeed obeys the boundary conditions.