Assignment 3

Due date: Wednesday, 16 November (8pm)

Ex 16

A field $u(x,t)$ has kinetic energy T and potential energy V, where

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \; \frac{1}{2} u_t^2 \; , \\ V &= \int_{-\infty}^{+\infty} dx \; \left[\frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] \; , \end{aligned} \tag{2.1}$$

and a and $\lambda > 0$ are (real) constants.[1] The equation of motion for u is

$$u_{tt} - u_{xx} + 2\lambda u (u^2 - a^2) = 0 \, .$$

1. If u is to have finite energy, what boundary conditions must be imposed on u, u_x and u_t at $x = \pm\infty$? [5 marks]

   SOLUTION:

   The energy $E = T + V$ is the integral of a sum of squares, all of which must vanish at spatial infinity to ensure that E is finite. So

   $$u_t, \; u_x, \; u^2 - a^2 \to 0 \qquad \text{as } x \to \pm\infty \; .$$

   In particular, u must tend to $\pm a$.

2. Find the general travelling-wave solution(s) to the equation of motion, consistent with the boundary conditions found in part 1. Compute the total energy $E = T + V$ for these solutions. For which velocity do the solutions have the lowest energy?
   (The list of integrals at the end of the problem sheet on DUO might help.) [25 marks]

[1]This is a version of the "ϕ^4 theory". It's called like that because the scalar potential is quartic, and the field u is usually called ϕ.

SOLUTION:

There are four options for the boundary conditions, corresponding to the two possible limits of u as $x \to -\infty$ and as $x \to +\infty$. The corresponding travelling wave solutions are

$u(-\infty,t)$	$u(+\infty,t)$	$u(x,t) =$
$-a$	$-a$	$-a$
$+a$	$+a$	$+a$
$-a$	$+a$	$+a\ \tanh[a\sqrt{\lambda}\gamma(x-x_0-vt)]$
$+a$	$-a$	$-a\ \tanh[a\sqrt{\lambda}\gamma(x-x_0-vt)]$

The nontrivial travelling wave solutions

$$u(x,t) = \pm a\ \tanh[a\sqrt{\lambda}\gamma(x-x_0-vt)]$$

are obtained as in **Ex 13.3**, the only difference being that here $a, \lambda \neq 1$. The dependence on a and λ can be recovered by replacing $(u,x,t) \mapsto (au, a\sqrt{\lambda}x, a\sqrt{\lambda}t)$ in the solution found in **Ex 13.3**.

The constant solutions have zero energy. For the travelling wave solutions

$$\begin{aligned} u_t^2 &= a^4\lambda\gamma^2v^2\ \mathrm{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)] \\ u_x^2 &= a^4\lambda\gamma^2\ \mathrm{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)] \\ \lambda(u^2-a^2)^2 &= a^4\lambda\ \mathrm{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)]\ , \end{aligned}$$

where I used

$$\frac{d}{dy}\tanh(y) = \mathrm{sech}^2(y)\ , \qquad \tanh^2(y) - 1 = -\mathrm{sech}^2(y)\ .$$

So the energy of the travelling wave solutions is

$$\begin{aligned} E &= \frac{1}{2}\int_{-\infty}^{+\infty} dx\ \left[u_t^2 + u_x^2 + \lambda(u^2-a^2)\right] \\ &= \frac{\lambda a^4}{2}(1+\gamma^2(1+v^2))\int_{-\infty}^{+\infty} dx\ \mathrm{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)] \\ &= \frac{\sqrt{\lambda}a^3}{2\gamma}\cdot\underbrace{\frac{1-v^2+1+v^2}{1-v^2}}_{=2\gamma^2}\cdot\underbrace{\int_{-\infty}^{+\infty} dy\ \mathrm{sech}^4(y)}_{=4/3} = \frac{4}{3}\sqrt{\lambda}a^3\cdot\gamma\ , \end{aligned}$$

where $\gamma = 1/\sqrt{1-v^2}$ and I used the change of variable $y = a\sqrt{\lambda}\gamma(x-x_0-vt)$ as well as the integral

$$\int_{-\infty}^{+\infty} dy\ \mathrm{sech}^4(y) = \frac{4}{3}$$

given at the end of the problem sheet. The travelling waves with the least energy are the static solutions with $v=0$, hence $\gamma=1$, which have energy $\frac{4}{3}\sqrt{\lambda}a^3$.

3. One of the possible boundary conditions for part 1 implies that u is a kink, with $[u(x)]_{x=-\infty}^{x=+\infty} = 2a$. Use the Bogomol'nyi argument to show that the total energy $E = T+V$ of that configuration is bounded from below by $C\sqrt{\lambda}a^3$, where C is a constant that you should determine, and find the solution u which saturates this bound. Verify that this solution agrees with the lowest-energy solution of part 2. [20 marks]

SOLUTION:

$$
\begin{aligned}
E &= \frac{1}{2}\int_{-\infty}^{+\infty} dx \ \left[u_t^2 + u_x^2 + \lambda(u^2-a^2)\right] \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} dx \ \left[u_t^2 + (u_x \mp \sqrt{\lambda}(u^2-a^2))^2 \pm 2\sqrt{\lambda}(u^2-a^2)u_x\right] \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} dx \ \left[u_t^2 + (u_x \mp \sqrt{\lambda}(u^2-a^2))^2\right] \pm \sqrt{\lambda}\left[\frac{u^3}{3} - a^2 u\right]_{-\infty}^{+\infty} \\
&\geq \pm\sqrt{\lambda}\left[\frac{u^3}{3} - a^2 u\right]_{-\infty}^{+\infty}
\end{aligned}
$$

For the "kink", $\lim_{x\to\pm\infty} u = \pm a$ (correlated signs: $+$ with $+$, $-$ with $-$), therefore

$$
E \geq \pm 2\sqrt{\lambda}\left(\frac{a^3}{3} - a^3\right) = \mp\frac{4}{3}\sqrt{\lambda}a^3.
$$

a and λ are positive, so the stronger lower bound for the energy is $E \geq \frac{4}{3}\sqrt{\lambda}a^3$, which is obtained by picking the lower sign. So $C = 4/3$.

The solution which saturates the lower bound is static $u_t = 0$ and satisfies the Bogomol'nyi equation

$$
u_x = -\sqrt{\lambda}(u^2 - a^2) \ .
$$

We can solve the equation by separation of variables:

$$
\begin{aligned}
\int \frac{du}{a^2 - u^2} &= \sqrt{\lambda}\int dx \\
\frac{1}{a}\text{arctanh}\left(\frac{u}{a}\right) &= \sqrt{\lambda}(x - x_0) \\
u(x,t) &= a \ \tanh[a\sqrt{\lambda}(x - x_0)] \ ,
\end{aligned}
$$

which is nothing but the static kink found in part 1 as a travelling wave solution with $v = 0$. [**Check**: indeed the Bogomol'nyi bound for the energy coincides with the lower bound for the energy of kink and antikink (travelling wave) solutions found in part 1.]

Ex 21

Consider the modified KdV (or mKdV) equation that you studied in Ex. 13.1, namely

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

with the boundary conditions

$$u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty \ .$$

1. Find three conserved charges for the mKdV equation which involve u, u^2 and u^4 respectively. [25 marks]

SOLUTION:

- The mKdV equation can be written as a continuity equation

$$u_t + (2u^3 + u_{xx})_x = 0 \ ,$$

where we identify the charge density $\rho_1 = u$ and the current density $j_1 = 2u^3 + u_{xx}$. The BC's imply that $j \to 0$ as $x \to \pm\infty$, so the charge

$$Q_1 = \int_{-\infty}^{+\infty} dx \ u$$

is conserved.

- Now we try $\rho_2 = u^2$. To see if it satisfies a continuity equation $(\rho_2)_t + (j_2)_x = 0$ with a suitable current j_2, let's calculate

$$\begin{aligned}(u^2)_t = 2uu_t &\underset{\text{mKdV}}{=} -12u^3 u_x - 2uu_{xxx} = (-3u^4 - 2uu_{xx})_x + 2u_x u_{xx} \\ &= -(3u^4 + 2uu_{xx} - u_x^2)_x \ .\end{aligned}$$

We identify $j_2 = 3u^4 + 2uu_{xx} - u_x^2$, which has the same limit (equal to zero) as $x \to \pm\infty$. [**NOTE:** It is fine to drop x-derivatives of functions which have the same limits at $\pm\infty$ as I did in lectures. Here I keep track of the current even though we only care that it has the same limits at $\pm\infty$.]
Therefore the charge

$$Q_2 = \int_{-\infty}^{+\infty} dx \ u^2$$

is conserved.

- Now we try $\rho_4 = u^4$, and calculate

$$
\begin{aligned}
(u^4)_t &= 4u^3 u_t \underset{\text{mKdV}}{=} -24u^5 u_x - 4u^3 u_{xxx} = (-4u^6 - 4u^3 u_{xx})_x + 12u^2 u_x u_{xx} \\
&\underset{\text{mKdV}}{=} -(4u^6 + 4u^3 u_{xx})_x - 2(u_t + u_{xxx})u_{xx} \\
&= -(4u^6 + 4u^3 u_{xx} + 2u_t u_x + u_{xx}^2)_x + 2u_{tx} u_x \\
&= -(4u^6 + 4u^3 u_{xx} + 2u_t u_x + u_{xx}^2)_x + (u_x^2)_t \ .
\end{aligned}
$$

The right-hand side is not an x-derivative, but we can bring the time derivative to the left-hand side to write this as the continuity equation

$$
(\underbrace{u^4 - u_x^2}_{=\rho_4})_t + (\underbrace{4u^6 + 4u^3 u_{xx} + 2u_t u_x + u_{xx}^2}_{=j_4})_x = 0.
$$

The BC's imply that $j_4 \to 0$ as $x \to \pm\infty$, so the charge

$$
Q_4 = \int_{-\infty}^{+\infty} dx \ \left(u^4 - u_x^2\right)
$$

is conserved.

2. Evaluate these conserved quantities for the two non-trivial travelling-wave solutions

$$
u(x,t) = \pm\sqrt{v} \ \mathrm{sech}\left[\sqrt{v}(x - x_0 - vt)\right]
$$

that you found in Ex 13.1.
(The list of integrals at the end of the problem sheet on DUO might help.) [25 marks]

SOLUTION:

First of all, note that we can shift the integration variable $x \mapsto x + x_0 + vt$ to get rid of the integration constant x_0 and of time t in the charges. Indeed, the charges are conserved, so they don't depend on time. So I will simply take

$$
u(x) = \pm v^{1/2} \ \mathrm{sech}(v^{1/2} x)
$$

in the following. We will also need the integrals

$$
I_1 = \int_{-\infty}^{+\infty} dy \, \mathrm{sech}(y) = \pi \ , \quad I_2 = \int_{-\infty}^{+\infty} dy \, \mathrm{sech}^2(y) = 2 \ , \quad I_4 = \int_{-\infty}^{+\infty} dy \, \mathrm{sech}^4(y) = \frac{4}{3} \ ,
$$

which can be extracted from the table of integrals at the end of the problem sheet.

We easily calculate

$$
\begin{aligned}
Q_1 &= \pm v^{1/2} \int_{-\infty}^{+\infty} dx \ \mathrm{sech}(v^{1/2} x) = \pm \int_{-\infty}^{+\infty} dy \ \mathrm{sech}(y) = \pm\pi \ , \\
Q_2 &= v \int_{-\infty}^{+\infty} dx \ \mathrm{sech}^2(v^{1/2} x) = v^{1/2} \int_{-\infty}^{+\infty} dy \ \mathrm{sech}^2(y) = 2v^{1/2} \ ,
\end{aligned}
$$

where the second equality in both lines follows from setting $y = v^{1/2}x$. Note that the measure changes: $dx = v^{-1/2}dy$. This is a common source of errors.

For Q_4, we first calculate

$$u_x = \mp v \sinh(v^{1/2}x) \cdot \mathrm{sech}^2(v^{1/2}x) \ ,$$

then

$$\begin{aligned}
Q_4 &= v^2 \int_{-\infty}^{+\infty} dx \ \mathrm{sech}^4(v^{1/2}x) \cdot (1 - \sinh^2(v^{1/2}x)) \\
&= v^{3/2} \int_{-\infty}^{+\infty} dy \ \mathrm{sech}^4(y) \cdot (2 - \cosh^2(y)) \\
&= v^{3/2} \int_{-\infty}^{+\infty} dy \ (2\mathrm{sech}^4(y) - \mathrm{sech}^2(y)) \\
&= v^{3/2}(2I_4 - I_2) = v^{3/2}\left(\frac{8}{3} - 2\right) = \frac{2}{3}v^{3/2} \ .
\end{aligned}$$