Assignment 4

Due date: Monday, 6 December (9am)

Please select 2 of the following 3 exercises to be submitted and marked.

Ex 25

This is yet another question about the KdV equation $u_t + 6uu_x + u_{xxx} = 0$. This time we will focus on conservation laws and what they can teach us about the time evolution of the class of initial conditions that we experimented with in the first lecture.

1. Evaluate the first three KdV conserved charges

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_3 = \int_{-\infty}^{+\infty} dx\ \left(u^3 - \frac{1}{2}u_x^2\right) \tag{4.1}$$

for the initial state $u(x,0) = A\,\mathrm{sech}^2(Bx)$, where A and B are constants. [20 marks]

SOLUTION:

I will use the definite integrals (where $n \in \mathbb{N}$)

$$I_n := \int_{-\infty}^{+\infty} dy\ \mathrm{sech}^{2n}(y) = \frac{2^{2n-1}((n-1)!)^2}{(2n-1)!}$$
$$\Longrightarrow\ I_1 = 2, \quad I_2 = \frac{4}{3}\ , \quad I_3 = \frac{16}{15}\ , \quad I_4 = \frac{32}{35}\ , \quad I_5 = \frac{256}{315}\ , \quad \ldots$$

which are tabulated at the end of the problem sheet, as well as the derivative formula

$$\frac{d}{dy}\mathrm{sech}\ y = -\mathrm{sech}\ y \cdot \tanh y = -\mathrm{sech}^2 y \cdot \sinh y\ ,$$

which implies

$$\frac{d}{dy}\mathrm{sech}^2 y = -2\ \mathrm{sech}^3 y \cdot \sinh\ y$$
$$\left(\frac{d}{dy}\mathrm{sech}^2 y\right)^2 = 4\ \mathrm{sech}^6 y \cdot \sinh^2 y = 4\ \mathrm{sech}^6 y \cdot (\cosh^2 y - 1) = 4\ (\mathrm{sech}^4 y - \mathrm{sech}^6 y)\ .$$

We can then evaluate Q_1, Q_2, Q_3 at $t = 0$ by changing integration variable $y = Bx$:

$$\begin{aligned} Q_1 &= \frac{A}{B} I_1 = \frac{2A}{B} \\ Q_2 &= \frac{A^2}{B} I_2 = \frac{4A^2}{3B} \\ Q_3 &= \frac{A^3}{B} I_3 - 2A^2 B (I_2 - I_3) = \frac{8A^2(2A - B^2)}{15B} \; . \end{aligned}$$

2. The initial condition

$$u(x, 0) = N(N+1)\,\mathrm{sech}^2(x) \; , \tag{4.2}$$

where N is an integer, is known to evolve at late times into N well-separated solitons, with velocities $4k^2$, $k = 1 \ldots N$. So for $t \to +\infty$, this solution approaches the sum of N single well-separated solitons

$$u(x, t) \approx \sum_{k=1}^{N} 2k^2\,\mathrm{sech}^2\big[k(x - x_k - 4k^2 t)\big] \; . \tag{4.3}$$

Since Q_1, Q_2 and Q_3 are conserved, their values at $t = 0$ and $t \to +\infty$ must be equal. Use this fact to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers. [30 marks]

SOLUTION:

To calculate the conserved charges Q_1, Q_2, Q_3 at the initial time $t = 0$, we can just set $A = N(N+1)$ and $B = 1$ in the results of part 1:

$$\begin{aligned} Q_1 &= 2N(N+1) \\ Q_2 &= \frac{4}{3} N^2 (N+1)^2 \\ Q_3 &= \frac{8}{15} N^2 (N+1)^2 (2N^2 + 2N - 1) \; . \end{aligned}$$

To calculate the conserved charges Q_1, Q_2, Q_3 at late times $t \to +\infty$, we use the fact that the N solitons are well-separated, that is, they are separated by much larger distances than the widths of the solitons. So

$$u \approx \sum_{k=1}^{N} u_{k,x_k} \; , \qquad u^2 \approx \sum_{k=1}^{N} u_{k,x_k}^2 \; , \qquad u^3 \approx \sum_{k=1}^{N} u_{k,x_k}^3 \; , \qquad u_x^2 \approx \sum_{k=1}^{N} (\partial_x u_{k,x_k})^2$$

where

$$u_{k,x_k}(x, t) = 2k^2\,\mathrm{sech}^2\big[k(x - x_k - 4k^2 t)\big] \; .$$

The cross terms are negligible because the soliton solutions u_{k,x_k} tend to zero exponentially fast away from their centres, and the solitons are assumed to be well-separated. So the contributions of the N well-separated solitons simply add up. Comparing with part 1, the k-th soliton u_{k,x_k} has $A_k = 2k^2$ and $B_k = k$ (the shift of x by $x_k + 4k^2 t$ is inconsequential for the calculation of the charges as it can be absorbed by a shift of the integration variable). So we find the charges

$$\begin{aligned}
Q_1 &= 2\sum_{k=1}^{N} \frac{A_k}{B_k} = 4\sum_{k=1}^{N} k \\
Q_2 &= \frac{4}{3}\sum_{k=1}^{N} \frac{A_k^2}{B_k} = \frac{16}{3}\sum_{k=1}^{N} k^3 \\
Q_3 &= \frac{8}{15}\sum_{k=1}^{N} \frac{A_k^2(2A_k - B_k^2)}{B_k} = \frac{32}{5}\sum_{k=1}^{N} k^5 \ .
\end{aligned}$$

Equating the $t = 0$ expressions and the $t \to +\infty$ expressions for the conserved charges we find

$$\sum_{k=1}^{N} k = \frac{1}{2}N(N{+}1)\ , \quad \sum_{k=1}^{N} k^3 = \frac{1}{4}N^2(N{+}1)^2\ , \quad \sum_{k=1}^{N} k^5 = \frac{1}{12}N^2(N{+}1)^2(2N^2{+}2N{-}1)\ .$$

Ex 26

1. Show that the pair of equations

$$\begin{aligned}
(u - v)_+ &= \sqrt{2}\, e^{(u+v)/2} \\
(u + v)_- &= \sqrt{2}\, e^{(u-v)/2}
\end{aligned} \tag{5.1}$$

provides a Bäcklund transformation linking solutions of $v_{+-} = 0$ (the wave equation in light-cone coordinates) to those of $u_{+-} = e^u$ (the Liouville equation). [15 marks]

SOLUTION:

Cross-differentiate (5.1) to get

$$\begin{aligned}
(u - v)_{+-} &= \frac{1}{\sqrt{2}}\, e^{(u+v)/2}(u + v)_- = e^{(u+v)/2}e^{(u-v)/2} = e^u \\
(u + v)_{-+} &= \frac{1}{\sqrt{2}}\, e^{(u-v)/2}(u - v)_+ = e^{(u-v)/2}e^{(u+v)/2} = e^u \ .
\end{aligned}$$

Taking sum and difference we obtain

$$\begin{aligned}
u_{+-} &= e^u \qquad \text{(Liouville eqn)} \\
v_{+-} &= 0 \qquad \text{(the wave eqn)}
\end{aligned}$$

2. Starting from d'Alembert's general solution $v = f(x^+) + g(x^-)$ of the wave equation, use the Bäcklund transform (5.1) to obtain the corresponding solutions of the Liouville equation for u.
[**Hint**: Set $u(x^+, x^-) = 2U(x^+, x^-) + f(x^+) - g(x^-)$. You might simplify the notation by setting $f(x^+) = \log(F'(x^+))$ and $g(x^-) = -\log(G'(x^-))$, where prime means first derivative.] [35 marks]

SOLUTION:

Substituting the general solution $v = f(x^+) + g(x^-)$ of the wave equation in the Bäcklund transform we obtain a system of two first order PDEs for a solution u of the Liouville equation:

$$\begin{cases} u_+ - f'(x^+) = \sqrt{2}\ e^{\left[u+f(x^+)+g(x^-)\right]/2} \\ u_- + g'(x^-) = \sqrt{2}\ e^{\left[u-f(x^+)-g(x^-)\right]/2} \end{cases}$$

The system simplifies if we make the substitution $u(x^+, x^-) = 2U(x^+, x^-) + f(x^+) - g(x^-)$ given in the hint:

$$\begin{cases} 2U_+ = \sqrt{2}\ e^{U+f(x^+)} \\ 2U_- = \sqrt{2}\ e^{U-g(x^-)} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} e^{-U}\ U_+ = \frac{1}{\sqrt{2}}\ e^{f(x^+)} \equiv \frac{1}{\sqrt{2}}\ F'(x^+) \\ e^{-U}\ U_- = \frac{1}{\sqrt{2}}\ e^{-g(x^-)} \equiv \frac{1}{\sqrt{2}}\ G'(x^-) \end{cases}$$

where in the last expression we used $f(x^+) = \log(F'(x^+))$ and $g(x^-) = -\log(G'(x^-))$ as suggested. This system can be integrated to get

$$-e^{-U} = \frac{1}{\sqrt{2}} \left(F(x^+) + G(x^-) + c\right)$$

where c is an integration constant. Taking the logarithm of the previous equation,

$$U = -\log\left[-\frac{1}{\sqrt{2}} \left(F(x^+) + G(x^-) + c\right)\right]$$

so

$$\begin{aligned} u &= 2U + \log\left[F'(x^+)G'(x^-)\right] \\ &= -2\log\left[-\frac{1}{\sqrt{2}} \left(F(x^+) + G(x^-) + c\right)\right] + \log\left[F'(x^+)G'(x^-)\right] \\ &= \log\frac{2F'(x^+)G'(x^-)}{\left(F(x^+) + G(x^-) + c\right)^2}\ . \end{aligned}$$

Ex 32

1. The argument of the arctangent in the sine-Gordon 2-soliton solution

$$u(x,t) = 4 \arctan\left(\mu\, \frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}}\right), \qquad \theta_i = \varepsilon_i \gamma_i (x - v_i t - \bar{x}_i)$$

where $\mu = (a_2 + a_1)/(a_2 - a_1)$, $v_i = (a_i^2 - 1)/(a_i^2 + 1)$, $\gamma_i = 1/\sqrt{1 - v_i^2}$, $\varepsilon_i = \text{sign}(a_i)$, and $\bar{x}_1$ and $\bar{x}_2$ are constants as in the lectures, is a continuous function of $x \in \mathbb{R}$. Show that, in particular, it is never infinite for finite x. What does this imply about the range of u? [**Hint**: consider the graph of $\tan u/4$.] [20 marks]

SOLUTION:

Here we are not considering the breather solution, but rather a genuine 2-soliton solution. θ_1 and θ_2 are real, therefore $1 + e^{\theta_1+\theta_2} > 0$ (in fact > 1), which implies that the argument of the arctangent is never infinite. Hence

$$\frac{1}{4} u(x,t) \neq \frac{\pi}{2} \mod \pi \qquad \Longleftrightarrow \qquad u(x,t) \neq 2\pi \mod 4\pi$$

for all $x \in \mathbb{R}$ at fixed time t. Therefore if

$$u(x,t) \in (-2\pi, +2\pi) + 4n\pi$$

for some integer n at some finite value of x (at fixed t), then by continuity

$$u(x,t) \in (-2\pi, +2\pi) + 4n\pi$$

with the same integer n for all finite values of x (at fixed t). In fact the previous argument applies for all values of x and t, since u is a continuous function of both x and t.

2. By taking the limits of this function as $x \to \pm\infty$ (with $t = \bar{x}_1 = \bar{x}_2 = 0$ for simplicity), show that the topological charge of the two-soliton solution is 0 if $\text{sign}(a_1) = \text{sign}(a_2)$, and ± 2 if $\text{sign}(a_1) = -\text{sign}(a_2)$, in units where the topological charge of a kink is 1. [30 marks]

SOLUTION:

If $t = \bar{x}_1 = \bar{x}_2 = 0$, the 2-soliton solution becomes

$$\tan \frac{u}{4} = \mu\, \frac{e^{\varepsilon_1 \gamma_1 x} - e^{\varepsilon_2 \gamma_2 x}}{1 + e^{(\varepsilon_1 \gamma_1 + \varepsilon_2 \gamma_2) x}} \tag{5.2}$$

where $\varepsilon_i = \text{sign}(a_i)$. We need to consider two cases: either a_1 and a_2 have the same sign, or they have opposite sign.

(a) $\underline{\varepsilon_1 = \varepsilon_2 \quad \Leftrightarrow \quad \text{sign}(a_1) = \text{sign}(a_2) :}$

Both exponentials $e^{\varepsilon_1\gamma_1 x}$ and $e^{\varepsilon_2\gamma_2 x}$ tend to zero or to infinity in any given limit $x \to \pm\infty$. In the former case the numerator of the RHS of (5.2) tends to zero and the denominator tends to 1. In the latter case the numerator and the denominator diverge, but the denominator diverges faster because of the sum in the exponent. In either case

$$\lim_{|x|\to+\infty} \tan\frac{u}{4} = 0 \ .$$

Choosing for definiteness the branch of the arctangent such that $u \to 0$ as $x \to -\infty$, it follows from the previous limit and the continuity argument in part 1 that $u \to 0$ as $x \to +\infty$ as well. Therefore the topological charge of the solution is

$$\frac{1}{2\pi}\left(\lim_{x\to+\infty} u(x,t) - \lim_{x\to-\infty} u(x,t)\right) = 0 \ .$$

(b) $\underline{\varepsilon_1 = -\varepsilon_2 \quad \Leftrightarrow \quad \text{sign}(a_1) = -\text{sign}(a_2) :}$

In this case in any given limit $x \to \pm\infty$ one of the two exponentials $e^{\varepsilon_1\gamma_1 x}$ and $e^{\varepsilon_2\gamma_2 x}$ tends to zero and the other exponential tends to infinity. Then the numerator diverges faster than the denominator (if that diverges at all). Hence

$$\lim_{|x|\to+\infty} \tan\frac{u}{4} = \pm\infty \ ,$$

with opposite signs for $x \to \pm\infty$. Choosing for definiteness the branch of the arctangent such that $u \to 0$ as $x \to -\infty$, it follows from the previous limit and the continuity argument in part 1 that $u \to 0$ as $x \to +\infty$ as well. Therefore the topological charge of the solution is

$$\frac{1}{2\pi}\left(\lim_{x\to+\infty} u(x,t) - \lim_{x\to-\infty} u(x,t)\right) = \frac{\pm 4\pi}{2\pi} = \pm 2 \ .$$