Assignment 5

Due date: Friday, 26 January (8pm)

Ex 44

It is given that the system of Hirota equations

$$\begin{cases} (D_x^2 - D_t^2 - 1)(f \cdot g) = 0 \\ (D_x^2 - D_t^2)(f \cdot f) = (D_x^2 - D_t^2)(g \cdot g) \end{cases}$$

yields solutions $u = 4 \arctan(g/f)$ of the sine-Gordon equation. Let $\theta_i = \alpha_i x + \beta_i t + \gamma_i$, where $\alpha_i, \beta_i, \gamma_i$ are constants.

1. Take

$$f = 1 \ , \qquad g = \epsilon e^{\theta_1}$$

and work order by order in powers of ϵ to find the one-soliton solution of the sine-Gordon equation. [35 marks]

SOLUTION:

Let's write the system of Hirota equations as a power series in ϵ, using the shorthand $B = D_x^2 - D_t^2$ and bilinearity:

$$\begin{cases} 0 = (B-1)(f \cdot g) = (B-1)(1 \cdot 0) + \epsilon(B-1)(1 \cdot e^{\theta_1}) \\ 0 = B(1 \cdot 1) - \epsilon^2 B(e^{\theta_1} \cdot e^{\theta_1}) \end{cases}$$

Now we solve the system order by order.

- Order ϵ^0: we have the system

$$\begin{cases} 0 = (B-1)(1 \cdot 0) \\ 0 = B(1 \cdot 1) \end{cases} .$$

This is trivially satisfied, since $B(1 \cdot 0) = 1(1 \cdot 0) = B(1 \cdot 1) = 0$, using bilinearity and/or direct differentiation.

- Order ϵ^1: we have the equation
$$0 = (B-1)(1\cdot e^{\theta_1}) = (\partial_x^2 - \partial_t^2 - 1)e^{\theta_1} = (a_1^2 - b_1^2 - 1)e^{\theta_1}\ ,$$
so the parameters must satisfy $a_1^2 = b_1^2 + 1$.
- Order ϵ^2: we have the equation $B(e^{\theta_1}\cdot e^{\theta_1}) = 0$, which is trivially satisfied using Lemma 1 from the notes (or explicit differentiation).

2. Taking e^{θ_i} as in the solution of the previous part, repeat the exercise for
$$f = 1 + \epsilon^2 f_2\ , \qquad g = \epsilon(e^{\theta_1} + e^{\theta_2})\ ,$$
and check that the Hirota equations are satisfied to all orders in ϵ. [65 marks]

SOLUTION:

Subbing in
$$f = 1 + \epsilon^2 f_2\ , \qquad g = \epsilon g_1 \equiv \epsilon(e^{\theta_1} + e^{\theta_2}),$$
using some of the above results, bilinearity and the symmetry property of $B(h\cdot k) = B(k\cdot h)$, we find the system
$$\begin{cases} 0 = (B-1)((1+\epsilon^2 f_2)\cdot(\epsilon g_1)) = \epsilon(B-1)(1\cdot g_1) + \epsilon^3(B-1)(f_2\cdot g_1) \\ 0 = B((1+\epsilon^2 f_2)\cdot(1+\epsilon^2 f_2)) - B((\epsilon g_1)\cdot(\epsilon g_1)) = \epsilon^2\left[2B(f_2\cdot 1) - B(g_1\cdot g_1)\right] + \epsilon^4 B(f_2\cdot f_2)\end{cases}$$
We need to solve the system order by order. Note that we have a single equation at each order, since the first Hirota equation is odd in ϵ, while the second Hirota equation is even.

- Order ϵ^0: this is trivially satisfied.
- Order ϵ^1: the equation $(B-1)(1\cdot(e^{\theta_1}+e^{\theta_2})) = 0$ is satisfied using bilinearity and $a_i^2 = b_i^2 + 1$ (as in part a).
- Order ϵ^2: we have the equation
$$\begin{aligned} 0 &= 2B(f_2\cdot 1) - B((e^{\theta_1}+e^{\theta_2})\cdot(e^{\theta_1}+e^{\theta_2})) \\ &= 2(\partial_x^2 - \partial_t^2)f_2 - 2B(e^{\theta_1}\cdot e^{\theta_2}) \\ &= 2(\partial_x^2 - \partial_t^2)f_2 - 2\left[(a_2-a_1)^2 - (b_2-b_1)^2\right]e^{\theta_1+\theta_2}\ ,\end{aligned}$$
using bilinearity and Lemmata 1 and 2. (It's understood that $a_i^2 = b_i^2 + 1$.) This equation determines f_2. As for the 2-soliton solution of KdV, we can take $f_2 = Ae^{\theta_1+\theta_2}$ for a constant A to be determined. Subbing in the previous equation, we obtain
$$A\left[(a_1+a_2)^2 - (b_1+b_2)^2\right]e^{\theta_1+\theta_2} = \left[(a_2-a_1)^2 - (b_2-b_1)^2\right]e^{\theta_1+\theta_2}\ ,$$
which determines
$$A = \frac{(a_2-a_1)^2 - (b_2-b_1)^2}{(a_1+a_2)^2 - (b_1+b_2)^2}\ .$$
One can use $a_i^2 = b_i^2 + 1$ to simplify the result, but this is not necessary.

- <u>Order ϵ^3:</u> simplifying a factor of A, we have to check that the equation

  $$0 = (B-1)(e^{\theta_1+\theta_2} \cdot (e^{\theta_1} + e^{\theta_2})) = (B-1)(e^{\theta_1+\theta_2} \cdot e^{\theta_1}) + (B-1)(e^{\theta_1+\theta_2} \cdot e^{\theta_2})$$

  is satisfied, since there are no unknowns left. The two terms in the RHS vanish individually, as in the solution to problem 40. Let's check it explicitly for the first term (for the second term, swap 1 and 2):

  $$\left[(a_1 + a_2 - a_1)^2 - (b_1 + b_2 - b_1)^2 - 1\right] e^{2\theta_1+\theta_2} = [a_2^2 - b_2^2 - 1]e^{2\theta_1+\theta_2} = 0$$

  using $a_i^2 = b_i^2 + 1$.

- <u>Order ϵ^4:</u> we have

  $$B(f_2, f_2) = A^2 B(e^{\theta_1+\theta_2} \cdot e^{\theta_1+\theta_2}) = 0$$

  by Lemma 1 or by explicit differentiation.