Assignment 6

Due date: Friday, 9 February (8pm)

Ex 54

Consider the time independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x) \ ,$$

where the potential $V(x)$ is the sum of two delta functions:

$$V(x) = -a\delta(x) - b\delta(x-r) \ .$$

Taking $r > 0$, the solution $\psi(x)$ can be split into three pieces, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$, defined on $(-\infty, 0)$, $(0, r)$, and $(r, +\infty)$ respectively.

1. Write down the four matching conditions relating ψ_1, ψ_2 and ψ_3, and their derivatives, at $x = 0$ and $x = r$. [10 marks]

2. For a scattering solution describing waves incident from the left, ψ_1 and ψ_3 are given by
$$\psi_1(x) = e^{ikx} + R(k)\, e^{-ikx}, \quad \psi_3(x) = T(k)\, e^{ikx}.$$
Write down the general form of ψ_2, and then use the matching conditions found in part 1 to eliminate the unknowns and determine $R(k)$ and $T(k)$. [40 marks]

3. Show from the answer to part 2 that, for there to be a bound state pole at $k = i\mu$, μ must satisfy
$$e^{-2\mu r} = (1 - 2\mu/a)(1 - 2\mu/b) \ . \qquad (***)$$
[10 marks]

4. The solutions to (***) can be analysed using a graphical method. Show that:
   (a) if both a and b are negative, then there are no bound states;
   (b) if a and b have opposite signs, then there is at most one bound state, occurring when $a + b > rab$ (note: since a and b have opposite signs, rab is negative);
   (c) if a and b are positive, then the number of bound states is one if $rab \leq a + b$, and two otherwise.

   Sketch on the ab-plane the regions corresponding to zero, one and two bound states, and indicate the form of $\psi(x)$ for each of the two bound states found when $ab/(a+b) > r^{-1}$. [40 marks]