Assignment 6

Due date: Monday, 12 February (12 noon)

Ex 54

Consider the time independent Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x) \ ,$$

where the potential $V(x)$ is the sum of two delta functions:

$$V(x) = -a\delta(x) - b\delta(x-r) \ .$$

Taking $r > 0$, the solution $\psi(x)$ can be split into three pieces, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$, defined on $(-\infty, 0)$, $(0, r)$, and $(r, +\infty)$ respectively.

1. Write down the four matching conditions relating ψ_1, ψ_2 and ψ_3, and their derivatives, at $x = 0$ and $x = r$. [10 marks]

   SOLUTION:

   We have

   $$\psi(x) = \begin{cases} \psi_1(x)\,, & x < 0\,, \\ \psi_2(x)\,, & 0 < x < r\,, \\ \psi_3(x)\,, & x > r\,. \end{cases}$$

   With $\psi = \psi_1$, ψ_2 or ψ_3 as above, at $x = 0$ we have

   $$\psi_1(0^-) = \psi_2(0^+) \equiv \psi(0) \ , \quad \psi_2'(0^+) - \psi_1'(0^-) = -a\psi(0) \ ,$$

   while at $x = r$,

   $$\psi_2(r^-) = \psi_3(r^+) \equiv \psi(r) \ , \quad \psi_3'(r^+) - \psi_2'(r^-) = -b\psi(r) \ .$$

2. For a scattering solution describing waves incident from the left, ψ_1 and ψ_3 are given by

   $$\psi_1(x) = e^{ikx} + R(k)\, e^{-ikx}, \quad \psi_3(x) = T(k)\, e^{ikx}.$$

   Write down the general form of ψ_2, and then use the matching conditions found in part 1 to eliminate the unknowns and determine $R(k)$ and $T(k)$. [40 marks]

SOLUTION:

$$\psi(x) = \begin{cases} e^{ikx} + R(k)\, e^{-ikx}\,, & x < 0\,, \\ A(k)\, e^{ikx} + B(k)(k)\, e^{-ikx}\,, & 0 < x < r\,, \\ T(k)\, e^{ikx}\,, & x > r\,. \end{cases}$$

Imposing the matching conditions at $x = 0$,

$$1 + R(k) = A(k) + B(k)\ , \quad ik(A(k) - B(k)) - ik(1 - R(k)) = -a(1 + R(k))\,,$$

so

$$A(k) + B(k) = 1 + R(k)\,, \tag{α}$$

$$A(k) - B(k) = \left(1 + i\frac{a}{k}\right) - R(k)\left(1 - i\frac{a}{k}\right)\,. \tag{β}$$

Likewise, looking at $x = r$,

$$A(k)\, e^{ikr} + B(k)\, e^{-ikr} = T(k)\, e^{ikr}$$

$$ikT(k)\, e^{ikr} - ik(A(k)\, e^{ikr} - B(k)\, e^{-ikr}) = -bT(k)\, e^{ikr},$$

so

$$A(k)\, e^{ikr} + B(k)\, e^{-ikr} = T(k)\, e^{ikr} \tag{γ}$$

$$A(k)\, e^{ikr} - B(k)\, e^{-ikr} = T(k)\, e^{ikr}\left(1 - i\frac{b}{k}\right)\,. \tag{δ}$$

Solving these for $A(k)$ and $B(k)$,

$$(\gamma) + (\delta): \quad A(k) = \left(1 - i\frac{b}{2k}\right) T(k)\,; \quad (\gamma) - (\delta): \quad B(k) = i\frac{b}{2k}\, e^{2ikr}\, T(k)\,.$$

Thus (α) and (β) become:

$$1 + R(k) = \left(1 - i\frac{b}{2k} + i\frac{b}{2k}\, e^{2ikr}\right) T(k) \tag{α'}$$

$$\left(1 - i\frac{b}{2k} - i\frac{b}{2k}\, e^{2ikr}\right) T(k) = \cdots = 2 - \left(1 - i\frac{a}{k}\right)(1 + R(k)) \tag{β'}$$

and substituting for $1 + R(k)$ from (α') into (β') and solving for $T(k)$ yields

$$T(k) = \frac{4k^2/(ab)}{e^{2ikr} - (1 + i\frac{2k}{a})(1 + i\frac{2k}{b})} = \frac{4k^2}{abe^{2ikr} - (a + 2ik)(b + 2ik)}\,.$$

Finally we can use (α') once more to find

$$R(k) = \frac{1 + i\frac{2k}{b} - (1 - i\frac{2k}{a})e^{2ikr}}{e^{2ikr} - (1 + i\frac{2k}{a})(1 + i\frac{2k}{b})} = \frac{a(b + 2ik) - b(a - 2ik)e^{2ikr}}{abe^{2ikr} - (a + 2ik)(b + 2ik)}\,.$$

NOTE: problems like this can be solved more systematically using ‘transfer matrices’. Ask me about them if you are interested.

3. Show from the answer to part 2 that, for there to be a bound state pole at $k = i\mu$ (with $\mu > 0$), μ must satisfy

$$e^{-2\mu r} = (1 - 2\mu/a)(1 - 2\mu/b) \ . \qquad (***)$$

[10 marks]

SOLUTION:

Bound states occur at poles in $T(k)$ with $k = i\mu$, $\mu > 0$. This needs the denominator of the above formula for $T(k)|_{k=i\mu}$ to vanish, that is

$$e^{-2\mu r} = \left(1 - \frac{2\mu}{a}\right)\left(1 - \frac{2\mu}{b}\right)$$

as required.

4. The solutions to (***) can be analysed using a graphical method. Show that:

   (a) if both a and b are negative, then there are no bound states;

   (b) if a and b have opposite signs, then there is at most one bound state, occurring when $a + b > rab$ (note: since a and b have opposite signs, rab is negative);

   (c) if a and b are positive, then the number of bound states is one if $rab \le a + b$, and two otherwise.

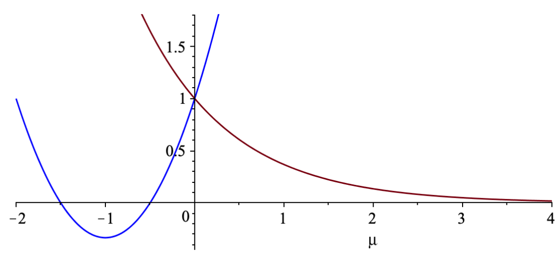
Sketch on the ab-plane the regions corresponding to zero, one and two bound states, and indicate the form of $\psi(x)$ for each of the two bound states found when $ab/(a+b) > r^{-1}$.

[40 marks]

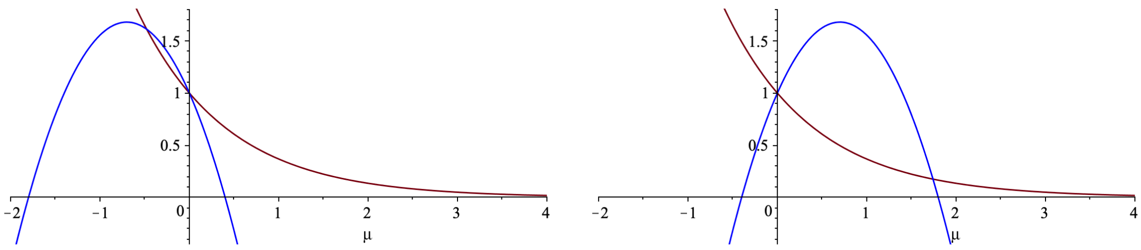
SOLUTION:

The LHS of (***), plotted in red below, is a simple decaying exponential, while the RHS (plotted in blue) is a quadratic in μ with zeros at $\mu = a/2$ and $\mu = b/2$. The two curves always intersect at $\mu = 0$; bound states will occur if there are further intersections with $\mu > 0$. Going case by case,

(a) For $a < 0$, $b < 0$, both zeros of the RHS are negative and so there are no intersections with $\mu > 0$:



(b) When a and b have opposite signs, there is one negative and one positive zero of the RHS, and the number of intersections with $\mu > 0$ will be either zero or one:



Which one occurs depends on the relative gradients of the LHS and RHS at $\mu = 0$. These gradients are

$$G_L = \frac{d}{d\mu} e^{-2\mu r}\Big|_{\mu=0} = -2r$$

and

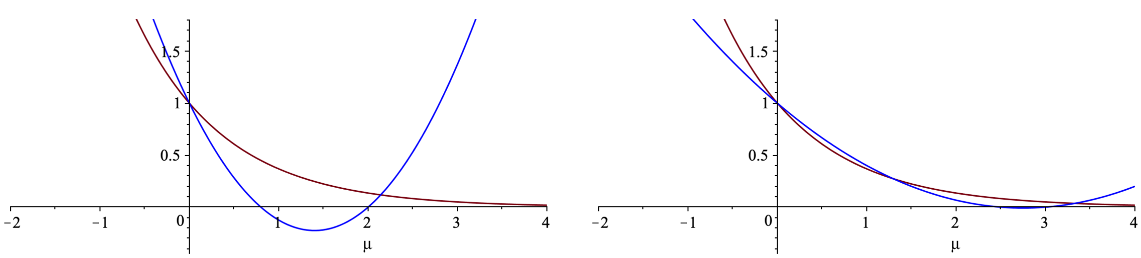
$$G_R = \frac{d}{d\mu}\left(1 - \frac{2\mu}{a}\right)\left(1 - \frac{2\mu}{b}\right)\Big|_{\mu=0} = -\frac{2}{a} - \frac{2}{b} = -2\frac{(a+b)}{ab}$$

and we are in the situation of the right-hand plot, with one bound state, when $G_L < G_R$, ie $-2r < -2(a+b)/(ab)$, or $r > (a+b)/(ab)$, or (noting that $ab < 0$ when rearranging the inequality)

$$a + b > rab\,,$$

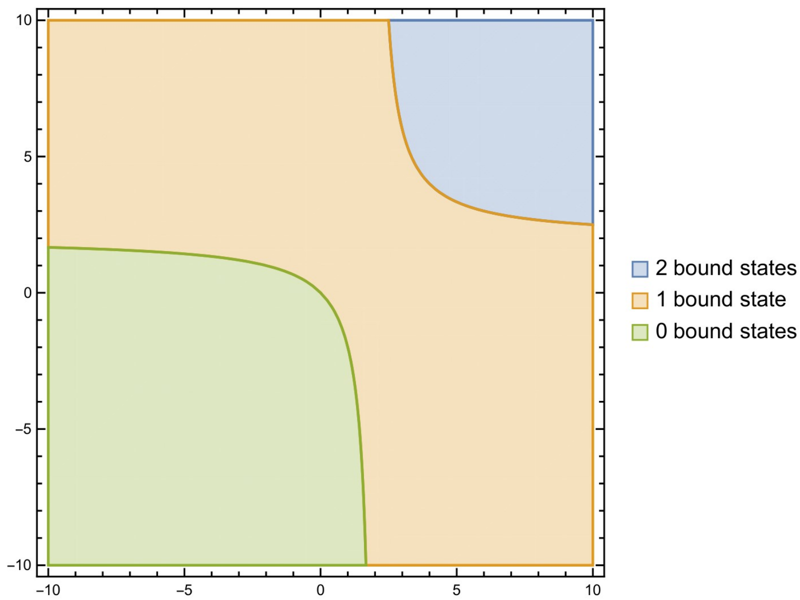
as required. Note that this should indeed be a strict inequality: when $a + b = rab$ the gradients at the origin are equal, and by considering the second derivatives (or otherwise) it can be shown that the only intersection is at $\mu = 0$, which does not give a bound state.

(c) When a and b are both positive, both zeros of the RHS are positive, and the number of intersections with $\mu > 0$ is either one or two:



Again, a comparison of the derivatives of the two curves at $\mu = 0$ determines which situation arises, and calculating as above shows that there is one bound state for $rab \leq a + b$ and two otherwise. Also as above, extra arguments need to be made when $rab = a + b$ to get the right answer in this case too.

For the last part, note that the transitions in the numbers of bound states occur on the curves $rab = a + b$, or $rab - a - b = 0$, or $r(a - 1/r)(b - 1/r) = 1/r$. On the a, b plane this is the hyperbola $b = 1/a$, but with the asymptotes shifted up and to the right, to $b = 1/r$ and $a = 1/r$. Here's a region plot in the (a, b)-plane for $r = 1/2$:



Finally, here's a rough sketch of the forms that $\psi(x)$ takes in the zone where there are two bound states: