Assignment 7

Due date: Monday, 26 February (12 noon)

Ex 56

Using the results stated in question 55 in the problem set, show that

$$V(x) = -2\mu^2 \text{sech}^2(\mu(x - x_0))$$

is an example of a reflectionless potential, for which $R(k) = 0$. By adjusting the normalisation of the wavefunction $\psi(x)$ correctly, find the transmission coefficient $T(k)$ for this potential. Verify that $|T(k)|^2 = 1$, consistent with the idea that for such a potential an incident particle must certainly be transmitted. [50 marks]

SOLUTION:

From question 55, we know that the solution to the Schroedinger problem is

$$\psi(x) = e^{ikx} (2k + iw(x)) \ ,$$

where $w(x)$ satisfies

$$w'(x) + \frac{1}{2}w^2(x) = 2\mu \ .$$

Substituting $w(x) = 2f'(x)/f(x)$ as suggested in the hint of question 55, we find

$$2\frac{f''f - (f')^2}{f} + 2\frac{(f')^2}{f} = 2\mu \quad \Longrightarrow \quad f'' = \mu f \ ,$$

which has general solution

$$f = Ae^{\mu x} + Be^{-\mu x}$$

for some constants A, B. Then

$$w = 2\frac{f'}{f} = 2\mu\frac{Ae^{\mu x} - Be^{-\mu x}}{Ae^{\mu x} + Be^{-\mu x}} = 2\mu \tanh(\mu(x - x_0)) \ ,$$

where in the last equality we traded A/B for x_0 (this is not necessary). Substituting $w = 2\mu \tanh(\mu(x - x_0))$ into the given equation for $\psi(x)$ and using the asymptotics of the hyperbolic tangent (or of the exponential) we have

$$\psi(x) = 2e^{ikx}(k + i\mu \tanh(\mu(x - x_0))) \approx \begin{cases} 2e^{ikx}(k - i\mu) \ , & x \to -\infty \\ 2e^{ikx}(k + i\mu) \ , & x \to +\infty \, . \end{cases}$$

Dividing through by $2(k - i\mu)$ gives us the correctly normalised scattering solution:

$$\psi_{\text{scattering}}(x) = e^{ikx}\,\frac{k + i\mu\tanh(\mu(x - x_0))}{k - i\mu} \approx \begin{cases} e^{ikx} & x \to -\infty\ , \\ \frac{k+i\mu}{k-i\mu}\,e^{ikx}\ , & x \to +\infty \end{cases}$$

from which we can read off that $R(k) = 0$ (so the potential is indeed reflectionless) and

$$T(k) = \frac{k + i\mu}{k - i\mu}\,.$$

Furthermore

$$|T(k)|^2 = \frac{|k + i\mu|^2}{|k - i\mu|^2} = \frac{k^2 + \mu^2}{k^2 + \mu^2} = 1$$

as expected, since k and μ are real.

Ex 60

Let $L(u) = D^2 + u(x,t)$ and $M(u) = \alpha D$ for some constant α, where $D = \frac{\partial}{\partial x}$.

1. Check that

$$L(u)_t = [M(u), L(u)] \quad \Longleftrightarrow \quad u_t = \alpha u_x\ .$$

[15 marks]

SOLUTION:
We have $L(u)_t = u_t$ (since the operator $D = \frac{\partial}{\partial x}$ does not depend on t), and

$$[M(u), L(u)] = \alpha[D, D^2 + u] = \alpha[D, u] = \alpha u_x\,.$$

Hence $L(u)_t = [M(u), L(u)] \iff u_t = \alpha u_x$ as required.

2. Let $\psi(x,0)$ be an eigenfunction of $L(u)$ at $t = 0$ with eigenvalue λ, so that

$$(D^2 + u(x,0))\psi(x,0) = \lambda\psi(x,0)\ .$$

If $u(x,t)$ evolves according to the equation of part 1, find an eigenfunction $\psi(x,t)$ for each later time t, with the same eigenvalue λ, so that

$$(D^2 + u(x,t))\psi(x,t) = \lambda\psi(x,t)\ .$$

Verify that $\psi(x,t)$ can be arranged to satisfy $\psi_t = M(u)\psi$. (You can assume that the eigenfunction is non-degenerate, namely that there is a single eigenfunction with that eigenvalue. This is the case both for bound state solutions and for scattering solutions.)

[35 marks]

SOLUTION:

If $u_t = \alpha u_x$ then $u(x,t) = f(x+\alpha t)$; matching to the initial condition at $t = 0$, $u(x,t) = u(x+\alpha t, 0)$. Now suppose that

$$(D^2 + u(x,0))\psi(x,0) = \lambda\psi(x,0)\,.$$

Replacing x by $x + \alpha t$ throughout,

$$(D^2 + u(x+\alpha t,0))\psi(x+\alpha t,0) = \lambda\psi(x+\alpha t,0)$$

but since $u(x,t) = u(x+\alpha t,0)$ this is the same as

$$(D^2 + u(x,t))\psi(x+\alpha t,0) = \lambda\psi(x+\alpha t,0)$$

and hence $(D^2 + u(x,t))\psi(x,t) = \lambda\psi(x,t)$ is solved by setting $\psi(x,t) = \psi(x+\alpha t,0)$. For this solution we have

$$\begin{aligned}\psi(x,t)_t &= \frac{\partial}{\partial t}\psi(x+\alpha t,0) = \frac{\partial(x+\alpha t)}{\partial t}\frac{\partial}{\partial x}\psi(x+\alpha t,0)\\ &= \alpha\frac{\partial}{\partial x}\psi(x+\alpha t,0) = \alpha\frac{\partial}{\partial x}\psi(x,t) = \alpha D\psi(x,t) = M(u)\psi(x,t)\end{aligned}$$

as required.