Assignment 8

Due date: Monday, 11 March (12 noon)

Ex 70

Consider the scattering data

$$S = \left\{ R(k) = \frac{a}{2ik - a}, \quad \left\{ \mu_1 = -\frac{a}{2}, \ c_1 = \sqrt{-\frac{a}{2}} \right\} \right\}$$

for the potential $V(x) = a\delta(x)$, with $a < 0$.

1. Calculate

$$F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} R(k) \, e^{-ik\xi} + \sum_{n=1}^{N} c_n^2 \, e^{\mu_n \xi} \ .$$

[**Hint:** close the integration contour of the k integral by adding an infinite arc in the upper or lower half of the complex plane for k, and use Cauchy's residue theorem. Look it up if you forgot it.] [50 marks]

SOLUTION:

For $a < 0$ there is a single bound state. From the scattering data

$$S = \left\{ R(k) = \frac{a}{2ik - a}, \left\{ \mu_1 = -\frac{a}{2}, c_1 = \sqrt{-\frac{a}{2}} \right\} \right\} \ ,$$

we find

$$F(\xi) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{a}{2ik - a} e^{-ik\xi} - \frac{a}{2} e^{-\frac{a}{2}\xi} \ .$$

The integrand has a pole at $k = -ia/2$, which has a positive imaginary part. As in the case $a > 0$ which was solved in Problems Class 7, we close the integration contour (originally the real k axis) in the complex plane by adding an infinite arc in the upper or lower half plane, picking the arc in such a way that it doesn't contribute to the integral, thanks to exponential damping.

- $\xi > 0$: we have

$$|e^{-ik\xi}| = e^{\xi \ \mathrm{Im}(k)} \ ,$$

therefore the integrand tends to zero exponentially fast along the lower infinite arc, where $\mathrm{Im}(k) \to -\infty$. Thus we find

$$F(\xi) = -\frac{a}{2} \oint_{C_-} \frac{dk}{2\pi i} \frac{e^{-ik\xi}}{k + i\frac{a}{2}} - \frac{a}{2} e^{-\frac{a}{2}\xi} \ ,$$

where C_- is the counterclockwise (and $-C_-$ the clockwise) oriented contour in the complex k plane consisting of the infinite arc in the lower half plane and the real line. Because the integrand is holomorphic (that, is it has no poles) in the region enclosed by C_-, the integral vanishes by Cauchy's residue theorem, and we obtain

$$F(\xi) = -\frac{a}{2} e^{-\frac{a}{2}\xi} \ .$$

- $\underline{\xi < 0}$: we have

$$|e^{-ik\xi}| = e^{\xi \ \mathrm{Im}(k)} \ ,$$

therefore the integrand tends to zero exponentially fast along the upper infinite arc, where $\mathrm{Im}(k) \to +\infty$. Adding (at no cost) this arc to the real k line we find that

$$F(\xi) = \frac{a}{2} \oint_{C_+} \frac{dk}{2\pi i} \frac{e^{-ik\xi}}{k + i\frac{a}{2}} - \frac{a}{2} e^{-\frac{a}{2}\xi} \ ,$$

where C_+ is the counterclockwise contour in the complex k plane consisting of the real line followed by the infinite arc in the upper half plane, until the contour is closed. Now the integration contour contains the single simple pole of the integrand (at $k = -ia/2$), and by Cauchy's residue theorem we find that

$$\begin{aligned} F(\xi) &= \frac{a}{2} \operatorname*{Res}_{k=-i\frac{a}{2}} \frac{e^{-ik\xi}}{k + i\frac{a}{2}} - \frac{a}{2} e^{-\frac{a}{2}\xi} \\ &= \frac{a}{2} e^{-\frac{a}{2}\xi} - \frac{a}{2} e^{-\frac{a}{2}\xi} = 0 \ . \end{aligned}$$

We can combine the results for $\xi > 0$ and $\xi < 0$ together in the single formula

$$F(\xi) = -\frac{a}{2} e^{-\frac{a}{2}\xi} \ \Theta(\xi) \ ,$$

where $\Theta(\xi)$ is the Heaviside step function.

[**Note:** The calculations above will get you full marks, but the careful student may complain that we didn't determine what happens if $\xi = 0$. We run into the problem that the integral in

$$\begin{aligned} F(0) &= -i\frac{a}{4\pi} \int_{-\infty}^{+\infty} dk \frac{1}{k + i\frac{a}{2}} - \frac{a}{2} \\ &= -i\frac{a}{4\pi} \int_{-\infty}^{+\infty} dk \left[\frac{k}{k^2 + \frac{a^2}{4}} - i\frac{a}{2} \frac{1}{k^2 + \frac{a^2}{4}} \right] - \frac{a}{2} \end{aligned}$$

is not absolutely convergent. If we regularize the imaginary part of the integral using Cauchy's principal value, then this imaginary part vanishes because we integrate an odd function over a symmetric domain. The real part of the integral is $a/4$, therefore we get $F(0) = -a/4$. This is consistent with taking $\Theta(0) = 1/2$, the average of the left-sided and right-sided limits of $\Theta(\xi)$ as $\xi \to 0$, in the formula for $F(\xi)$ above.]

2. Solve the Marchenko equation

$$K(x,z) + F(x+z) + \int_{-\infty}^{x} dy\, K(x,y)\, F(y+z) = 0$$

to determine the unknown function $K(x,z)$ for all $z \leq x$ (and set $K(x,z) = 0$ for $x < z$). [30 marks]

3. Show that

$$V(x) = 2\frac{d}{dx} \lim_{z \to x^-} K(x,z) \ .$$

[20 marks]

SOLUTION:

We found exactly the same result for $F(\xi)$ that we found for $a > 0$ in Problems Class 7 (as a function of a and ξ). The solution of parts 2 and 3 is then identical to the one for $a > 0$. See the handwritten notes for Problems Class 7 for the details.