REVISION LECTURE 1 - 25/4/2022

Ex 44 The complex field $u(x,t)$ obeys the equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0\ , \tag{7.1}$$

where $i = \sqrt{-1}$, and the boundary conditions

$$u, u_x, u_{xx} \to 0 \qquad \text{as} \quad x \to \pm\infty\ . \tag{7.2}$$

1. Show that the quantities

$$\begin{aligned} Q_1 &= \int_{-\infty}^{+\infty} dx\ |u|^2 \\ Q_2 &= \int_{-\infty}^{+\infty} dx\ \mathrm{Im}(\bar{u}u_x) \\ Q_3 &= \int_{-\infty}^{+\infty} dx\ \left(\frac{1}{2}|u_x|^2 + C|u|^4\right) \end{aligned} \tag{7.3}$$

are conserved provided that the constant C takes a value that you should find. (Here Im denotes the imaginary part and a bar denotes complex conjugation.)

$$u_t = i\left(\tfrac{1}{2}u_{xx} + |u|^2 u\right) \xrightarrow[|x|\to\infty]{} 0 \qquad (\text{similarly } u_{tx}, \dots)$$

Reminder: $\quad \rho_t + j_x = 0\ , \quad [j]_{-\infty}^{+\infty} = 0 \ \Rightarrow\ \frac{d}{dt}\int_{-\infty}^{+\infty} dx\ \rho = 0\ .$

- $(|u|^2)_t = \bar{u}u_t + \text{c.c.} = i\bar{u}\left(\tfrac{1}{2}u_{xx} + \underbrace{\cancel{|u|^2 u}}_{0}\right) + \text{c.c.} = \frac{i}{2}\bar{u}u_{xx} + \text{c.c.}$

$$= \frac{i}{2}(\bar{u}u_x)_x - \underbrace{\cancel{\frac{i}{2}|u_x|^2 + \text{c.c.}}}_{0} = \left(\frac{i}{2}\bar{u}u_x + \text{c.c.}\right)_x \equiv -j_x$$

BC $\Rightarrow\ j \to 0$ as $|x| \to \infty$.

$\Rightarrow \dfrac{d}{dt}Q_1 = 0\ .$

- $(\mathrm{Im}\,\bar{u}u_x)_t = \mathrm{Im}\,\bar{u}_t u_x + \mathrm{Im}\,\bar{u}u_{xt}$ "=" $\mathrm{Im}(\bar{u}_t u_x - \bar{u}_x u_t)$

$$\bar{u}_x u_t = i\left(\tfrac{1}{2}\bar{u}_x u_{xx} + |u|^2\bar{u}_x u\right)$$

$$u_x \bar{u}_t = -i\left(\tfrac{1}{2}\bar{u}_{xx}u_x + |u|^2\bar{u}u_x\right)$$

$$\bar{u}_t u_x - \bar{u}_x u_t = -i\left[\tfrac{1}{2}\left(\bar{u}_{xx}u_x + \bar{u}_x u_{xx}\right) + |u|^2\left(\bar{u}u_x + \bar{u}_x u\right)\right]$$

$$\bar{u}_t u_x - \bar{u}_x u_t = -i\left[\frac{1}{2}(\bar{u}_{xx}u_x + \bar{u}_x u_{xx}) + |u|^2(\bar{u}u_x + \bar{u}_x u)\right]$$

$$= -i\left[\frac{1}{2}|u_x|^2 + \frac{1}{2}|u|^4\right]_x \quad \text{"="} 0$$

$$\Rightarrow \frac{d}{dt} Q_2 = \frac{d}{dt}\int_{-\infty}^{+\infty} dx \, \mathrm{Im}(\bar{u}u_x) = 0 \, .$$

- $$Q_3 = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2}|u_x|^2 + C|u|^4\right)$$

$$\left(\frac{1}{2}|u_x|^2 + C|u|^4\right)_t = \frac{1}{2}\bar{u}_x u_{xt} + 2C|u|^2 \bar{u} u_t + \text{c.c.}$$

$$\text{"="} \left(-\frac{1}{2}\bar{u}_{xx} + 2C|u|^2\bar{u}\right)u_t + \text{c.c.}$$

$$= -i\left(+\frac{1}{2}\bar{u}_{xx} - 2C|u|^2\bar{u}\right)\left(\frac{1}{2}u_{xx} + |u|^2 u\right) + \text{c.c.}$$

$$= 0 \qquad \text{if} \quad \underline{\underline{C = -\frac{1}{2}}}$$

$$\Rightarrow \frac{d}{dt} Q_3 = \frac{d}{dt}\int_{-\infty}^{+\infty} dx \left(\frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4\right) = 0 \, .$$

2. Show that given a 'seed' solution $u(x,t)$ of equation (7.1),

$$u^{(v)}(x,t) := u(x-vt,t)e^{i(Ax+Bt)} \tag{7.4}$$

is also a solution for all $v \in \mathbb{R}$, provided that the real constants A and B depend on v in a way that you should find.

EoM: $\quad i u_t + \frac{1}{2} u_{xx} + |u|^2 u = 0 .$

meaning ∂ wrt 2^{nd} variable $\qquad\qquad$ meaning ∂ wrt 1^{st} variable

$$u^{(v)}_t = u_t(x-vt,t)\, e^{i(Ax+Bt)} - v\, u_x(x-vt,t)\, e^{i(Ax+Bt)} + iB\, u(x-vt,t)\, e^{i(Ax+Bt)}$$

$$= \left(u_t - v u_x + iBu\right) e^{i(Ax+Bt)}$$

$$u^{(v)}_x = \left(u_x + iAu\right) e^{i(Ax+Bt)}$$

$$u^{(v)}_{xx} = \left(u_{xx} + 2iAu_x - A^2 u\right) e^{i(Ax+Bt)}$$

So we need

$$0 = e^{i(Ax+Bt)}\left[i\left(u_t - vu_x + iBu\right) + \frac{1}{2}\left(u_{xx} + 2iAu_x - A^2u\right) + |u|^2u\right]$$

$$\Rightarrow \quad 0 = \cancel{-\frac{1}{2}u_{xx}} \cancel{-|u|^2u} - ivu_x - Bu \cancel{+\frac{1}{2}u_{xx}} + iAu_x - \frac{1}{2}A^2u \cancel{+|u|^2u}$$

$$= iu_x(A-v) - u\left(B + \frac{1}{2}A^2\right) \qquad \forall\, u, u_x \text{ (indep.)}$$

$$\Rightarrow \quad \underline{A = v}\,, \quad \underline{B = -\frac{1}{2}A^2 = -\frac{1}{2}v^2}\,.$$

$$u^{(v)}(x,t) = u(x-vt,t)\, e^{i\left(vx - \frac{1}{2}v^2 t\right)}$$

$$= u(x-vt,t)\, e^{iv(x-vt) + i\frac{v^2}{2}t} .$$

1-parameter family of solutions.

3. Determine the functional dependence of the conserved charges Q_1, Q_2, Q_3 in (7.3) on the parameter v that labels the one-parameter family of solution (7.4).

$$u^{(v)}(x,t) = u(x-vt,t)\, e^{iv(x-vt) + i\frac{v^2}{2}t}$$

- $$Q_1^{(v)} = \int_{-\infty}^{+\infty} dx\, |u^{(v)}(x,t)|^2 = \int dx\, |u(x-vt,t)|^2 = \int dx\, |u(x,t)|^2 = Q_1$$

- $$Q_2^{(v)} = \int dx\, \mathrm{Im}\left(\bar{u}^{(v)} u_x^{(v)}\right) = \int dx\, \mathrm{Im}\left[\bar{u}(u_x + ivu)\right] = \int dx \left[\mathrm{Im}(\bar{u}u_x) + v|u|^2\right]$$
$$= Q_2 + v\,Q_1 \,.$$

- $$Q_3^{(v)} = \int dx \left[\frac{1}{2}|u_x^{(v)}|^2 - \frac{1}{2}|u^{(v)}|^4\right] = \int dx \left(\frac{1}{2}|u_x + ivu|^2 - \underline{\frac{1}{2}|u|^4}\right)$$

$$|u_x + ivu|^2 = |u_x|^2 + \left(iv\,\bar{u}_x u - iv\,\bar{u}u_x\right) + v^2|u|^2$$
$$= \underline{|u_x|^2} + \underline{2v\,\mathrm{Im}(\bar{u}u_x)} \quad \underline{+\, v^2|u|^2}$$

$$Q_3^{(v)} = \underline{Q_3} + \underline{v\,Q_2} + \underline{\frac{v^2}{2}\,Q_1} \,.$$

4. Find all solutions of the form

$$u(x,t) = \underline{\rho(x)}\,\underline{e^{i\varphi(t)}} \tag{7.5}$$

of equation (7.1) with boundary conditions (7.2), where ρ are φ are real and $u(x,0)$ is a real even function of x. [You can use the integrals at the end of the problem sheet.] Apply the method of part 2 to this seed solution to find the associated one-parameter family of solutions $u^{(v)}(x,t)$.

$$\varphi(0)=0\,, \qquad \rho(x) = \rho(-x)\,.$$

EoM: $\quad i u_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$

$$0 = e^{i\varphi}\left[-\rho\dot\varphi + \frac{1}{2}\rho'' + \rho^3\right]$$

$$\Rightarrow \quad \underbrace{\dot\varphi}_{\text{fn of t only}} = \underbrace{\frac{1}{2}\frac{\rho''}{\rho} + \rho^2}_{\text{fn of x only}} = \alpha = \text{const.}$$

$$\begin{cases} \varphi(t) = \varphi(0) + \alpha t = \alpha t \\ \rho'' - 2\alpha\rho + 2\rho^3 = 0\,. \qquad (*) \end{cases}$$

$\int dx\, \rho' \times (*)$: $\quad \underline{\frac{1}{2}(\rho')^2} - \alpha\rho^2 + \underline{\frac{1}{2}\rho^4} = \beta = \text{const} = 0$ by BC

(Need $\alpha>0$.)

$$\rho' = \pm\sqrt{2\alpha\rho^2 - \rho^4} = \pm\sqrt{2\alpha}\,\rho\sqrt{1-\frac{1}{2\alpha}\rho^2}$$

$$\int \frac{d\rho}{\rho\sqrt{1-\frac{1}{2\alpha}\rho^2}} = \pm\sqrt{2\alpha}\int dx = \pm\sqrt{2\alpha}\,(x-x_0)$$

$\rho = \sqrt{2\alpha}\, r$ $\quad \|$ $\qquad\qquad\qquad\qquad \|$

$$\int \frac{dr}{r\sqrt{1-r^2}} = -\,\text{arcsech}(r) = -\,\text{arcsech}\left(\frac{\rho}{\sqrt{2\alpha}}\right)$$

$$\Rightarrow \quad \rho(x) = \sqrt{2\alpha}\ \operatorname{sech}\left(\sqrt{2\alpha}\,(x-x_0)\right)$$

<u>even</u> iff $x_0 = 0$.

$$\rho(x) = \sqrt{2\alpha}\ \operatorname{sech}\left(\sqrt{2\alpha}\,x\right)$$

$$\Rightarrow \quad u(x,t) = \rho(x)\,e^{i\varphi(t)} = \sqrt{2\alpha}\ \operatorname{sech}\left(\sqrt{2\alpha}\,x\right)\,\underline{e^{i\alpha t}} \qquad (\alpha > 0)$$

$$u^{(v)}(x,t) = u(x-vt,t)\,e^{i\left[v\underline{(x-vt)} + \underline{\frac{v^2}{2}}t\right]}$$

$$= \sqrt{2\alpha}\ \operatorname{sech}\left[\sqrt{2\alpha}\,(x-vt)\right]\,e^{i\left[v(x-vt) + \left(\alpha + \frac{v^2}{2}\right)t\right]} \ ,$$