

# SOLUTIONS TO SELECTED EXERCISES

**Ex 1**

Sub  $u(x,t) = \frac{2}{\cosh^2(x-vt)}$  in KdV eqn  $u_t + 6uu_x + u_{xxx} = 0$  :

$$u_t = \frac{4v}{\cosh^3(x-vt)} \cdot \sinh(x-vt)$$

$$u_x = \frac{-4}{\cosh^3(x-vt)} \sinh(x-vt)$$

$$u_{xx} = \frac{12}{\cosh^4(x-vt)} \sinh^2(x-vt) - \frac{4}{\cosh^2(x-vt)} \overset{\substack{\uparrow \\ \cosh^2 x - \sinh^2 x = 1}}{=} \frac{8}{\cosh^2(x-vt)} - \frac{12}{\cosh^4(x-vt)}$$

$$u_{xxx} = \left[ \frac{-16}{\cosh^3(x-vt)} + \frac{48}{\cosh^5(x-vt)} \right] \sinh(x-vt)$$

$$\text{So KdV} \Rightarrow 0 = \frac{4 \sinh(x-vt)}{\cosh^3(x-vt)} \cdot \left[ v - \frac{48}{\cosh^2(x-vt)} - 4 + \frac{48}{\cosh^2(x-vt)} \right]$$

Therefore  $u = \frac{2}{\cosh^2(x-vt)}$  is a solution of KdV if (and only if)  $v=4$ .

**Ex 2**

1. Let  $u(x,t) = Ag(X,T)$ , where  $X=Bx$  and  $T=Ct$ .

Chain rule  $\Rightarrow u_t(x,t) = ACg_T(X,T)$ ,  $u_x(x,t) = ABg_X(X,T)$ ,  $u_{xxx}(x,t) = AB^3g_{XXX}(X,T)$

$$\begin{aligned} \Rightarrow u_t + 6uu_x + u_{xxx} &= ACg_T(X,T) + 6A^2Bg(X,T)g_X(X,T) + AB^3g_{XXX}(X,T) \\ &= AC \cdot \left[ g_T(X,T) + \frac{AB}{C} \cdot 6g(X,T)g_X(X,T) + \frac{B^3}{C}g_{XXX}(X,T) \right]. \end{aligned}$$

If  $g(X,T)$  solves KdV (in its variables  $X$  and  $T$ !), then  $u(x,t)$  solves KdV (in its variables  $x$  and  $t$ ) provided that  $\frac{AB}{C} = \frac{B^3}{C} = 1$ , because in that case

$$u_t + 6uu_x + u_{xxx} = AC \left[ g_T(X,T) + 6g(X,T)g_X(X,T) + g_{XXX}(X,T) \right] = 0.$$

$\frac{AB}{C} \overset{\uparrow}{=} \frac{B^3}{C} = 1$ 
 $\uparrow$   
KdV for  $g(X,T)$

$$\frac{AB}{C} = \frac{B^3}{C} = 1 \iff \underline{A=B^2, C=B^3}.$$

2.  $u(x,t) = g(x,t) \equiv \frac{2}{\cosh^2(x-4t)}$  1-soliton solution of KdV

$\Rightarrow u(x,t) = B^2 g(Bx, B^3t) = \frac{2B^2}{\cosh^2[B(x-4B^3t)]} \equiv u_B(x,t)$

1-parameter family of 1-soliton sol'n's (with parameter B)

↑  
label, not derivative!

3. (Height of  $u_B$ ) =  $2B^2$   
 (Velocity of  $u_B$ ) =  $4B^2 \Rightarrow$  Velocity = 2 Height.

The "width" is a measure of how the lump is concentrated in space ( $x$ ).

Since the dependence of  $u_B$  on  $x$  is only through  $Bx$ , we can say that (Width of  $u_B$ )  $\propto \frac{1}{B}$ . The precise proportionality factor depends on the precise definition of width that you might choose, but regardless of that choice

( $B > 0$  with no loss of generality)

$v \rightarrow 4v = B^2 v \Rightarrow \text{width} \rightarrow \frac{1}{B} \text{width} = \frac{\text{width}}{2}$ .

Don't worry if you were confused by part 3 because I didn't define the width unambiguously. My aim was to make you think about the meaning of width (and see how many of you would realize that the answer was largely independent of the def'n).

I might run a few more social experiments like this during the year, but not in the exam, I promise. And feel free to ask me questions about the homework if you think you need more information than is given in the text.

SUMMARY of this exercise: For 1-soliton solutions of KdV

Height  $\propto$  Velocity  $\propto$  Width<sup>-2</sup>.

### Ex 3

Direct method: check using the chain rule.

- $$v_t + 6\varepsilon v v_x + v_{xxx} = \frac{1}{\varepsilon} u_t + 6\varepsilon \frac{1}{\varepsilon^2} u u_x + \frac{1}{\varepsilon} u_{xxx} = \frac{1}{\varepsilon} (u_t + 6u u_x + u_{xxx}) = 0.$$
- $$w_t + 6w w_x + \varepsilon w_{xxx} = \varepsilon \left. \frac{\partial T}{\partial t} \frac{\partial u(x, T)}{\partial T} \right|_{T=\varepsilon t} + 6\varepsilon^2 u(x, T) u_x(x, T) \Big|_{T=\varepsilon t} + \varepsilon \cdot \varepsilon u_{xxx}(x, T) \Big|_{T=\varepsilon t} = \varepsilon^2 \left[ u_T(x, T) + 6u(x, T) u_x(x, T) + u_{xxx}(x, T) \right] \Big|_{T=\varepsilon t} = 0.$$

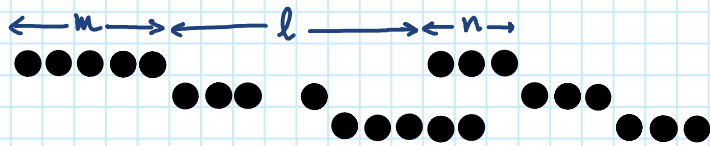
Alternatively, suppose you didn't know the final eqn for  $v, w$  and wanted to derive it. Express  $u$  in terms of  $v$  ( $w$ ) and sub in KdV eqn:

- $$u(x, t) = \varepsilon v(x, t): \quad \text{KdV} \Rightarrow \quad \varepsilon v_t + 6\varepsilon^2 v v_x + \varepsilon v_{xxx} = 0$$
$$\Rightarrow \quad v_t + 6\varepsilon v v_x + v_{xxx} = 0.$$

- $$u(x, \varepsilon t) = \frac{1}{\varepsilon} w(x, t): \quad \text{KdV for } u(x, \varepsilon t) \text{ is}$$

$$\begin{aligned} 0 &= \frac{\partial u(x, \varepsilon t)}{\partial(\varepsilon t)} + 6u(x, \varepsilon t) u_x(x, \varepsilon t) + u_{xxx}(x, \varepsilon t) \\ &\stackrel{\text{from } u(x, \varepsilon t) = \frac{1}{\varepsilon} w(x, t)}{=} \frac{1}{\varepsilon} \cdot \frac{1}{\varepsilon} w_t(x, t) + \frac{6}{\varepsilon^2} w(x, t) w_x(x, t) + \frac{1}{\varepsilon} w_{xxx}(x, t) \\ &\stackrel{\text{from } \frac{\partial t}{\partial(\varepsilon t)}}{=} \frac{1}{\varepsilon^2} (w_t + 6w w_x + \varepsilon w_{xxx}) \end{aligned}$$

**Ex 4**



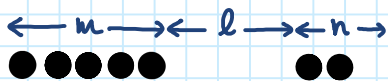
When the length  $m$  soliton is far enough to the left of the length  $n$  soliton (with  $m > n$ ), we can evolve the system forward by one unit of time, to reduce the separation  $l$ :

$$l \rightarrow l - m + n.$$

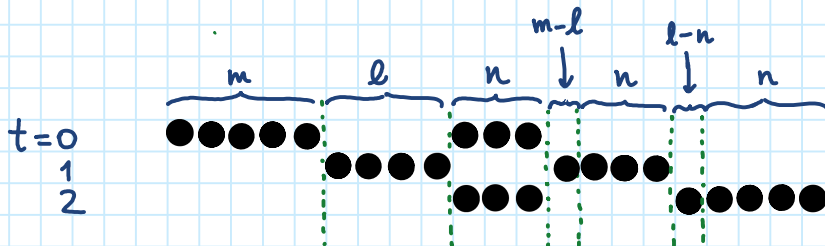
We can iterate this until the separation  $l$  reaches the range

$$n \leq l < m,$$

and take this as our starting point:



Let's evolve this configuration forward and see what happens to the two solitons. To cover all cases, we'll have to evolve the system by  $\Delta t = 2$ .



Make sure you understand how this follows from the ball and box rule. If you don't, ask me.

Unless  $n = l$ , at  $t = 1$  the system is in an intermediate configuration, still in the middle of the interaction. But already at  $t = 2$  the faster soliton has overtaken the slower, after which they keep travelling undisturbed to the right, with increasing separation.

To calculate the phase shifts of the two solitons, let's compare their positions at  $t = 2$  (after the interaction) with the positions they would have had at the same time had the other soliton not been there.

Compared to  $t=0$ , the length  $m$  soliton has moved to the right by

$$m+l+n+(m-l)+n = 2m+2n$$

boxes. It would have moved by  $2m$  boxes in the absence of an interaction, so its phase shift is  $+2n$ .

Compared to  $t=0$ , the length  $n$  soliton has moved to the right by  $0$  boxes.

It would have moved by  $2n$  boxes had no interaction taken place, therefore its phase shift is  $-2n$ .

### CONCLUSION :

The two solitons emerge from the interaction with the same lengths and velocities, but advanced/delayed by

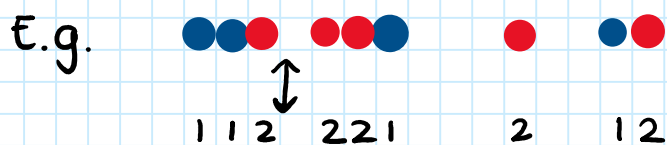
$$|\text{Phase shift}| = 2 \times (\text{Length of the slower soliton}) .$$

- What can go "wrong" if  $l < n$  ?

The starting point is in the middle of the interaction! See for instance  $t=1$  in the picture above. This is just an intermediate configuration, which can involve different objects than the solitons appearing in the far past/future.

## Ex 5

To make my life easier and aid memory, I will avoid colours and use instead "1" for the (blue) ball which is moved first and "2" for the (red) ball which is moved afterwards.



• Clearly:

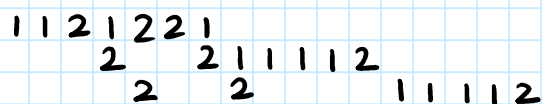
A row of  $n$  consecutive 1's is a speed  $n$  soliton

" " " " " 2's " " " " "

because if a single colour/type of ball is present, it evolves as in the single colour ball and box model.

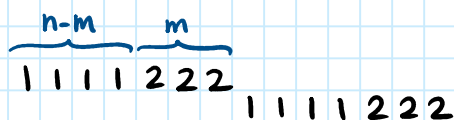


• Next, consider a sequence/row where 1's are separated by 2's, e.g.



Since 1's move first, after one unit of time all 1's will be consecutive, so this is not a soliton. In order to have a soliton, all the 1's must be consecutive!

• Consider then a row of  $m$  2's followed by  $n-m$  1's:



1's move first, translating by  $n$  boxes to the right, and 2's follow.

→ Speed  $n$  soliton.

This works for  $m=0,1,2,\dots,n \Rightarrow \underline{n+1}$  speed  $n$  solitons.

• What if we have  $m$  1's followed by  $n-m$  2's?

$$\begin{array}{c|c|c} \overbrace{2222111}^{n-m > m} & \overbrace{222111}^{n-m = m} & \overbrace{22111}^{n-m < m} \\ 2221112 & 222111 & 22 \quad 111 \end{array}$$

1's move first by  $m$  units. 2's follow:

- 1)  $n-m > m$  : at least one 2 overtakes the row of 1's.
- 2)  $n-m = m$  : final configuration = initial configuration.  
 $\rightarrow$  Speed  $m = \frac{n}{2}$  soliton (if  $n \in 2\mathbb{Z}$ ).
- 3)  $n-m < m$  : the row of 2's is left behind.

• Finally, what about

$$\overbrace{2222}^a \overbrace{111}^b \overbrace{2222}^c \quad ?$$

1's move first, leaving a gap between the two rows of 2's. The gap is (partially or completely) filled by some 2's from the left group, but anyway the 2's from the right group leave a gap of (at least)  $c$  empty boxes to the left of the row of 1's.

$$222 \underbrace{\quad\quad\quad}_c 111222222 \quad \rightarrow \text{Not a soliton!}$$

### SUMMARY:

•  $n$  is odd: there are  $n+1$  length  $n$  solitons, all of speed  $n$ .

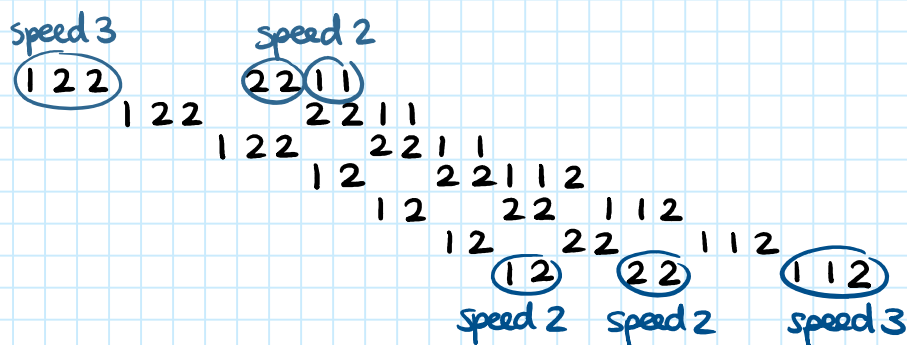
$$\overbrace{1111}^{n-m} \overbrace{222}^m \quad m = 0, 1, 2, \dots, n.$$

•  $n$  is even: in addition to the above, there's an extra length  $n$  soliton of speed  $\frac{n}{2} = m$ .

$$\overbrace{222}^m \overbrace{111}^m \\ 222111$$

In fact, this latter soliton is better thought of as the union of two consecutive length  $\frac{n}{2}$  and speed  $\frac{n}{2}$  solitons.

Indeed, consider the scattering ( $\rightarrow$  Ex 7)



Note that numbers (colours) are reshuffled, but lengths and speeds are unchanged.  
 The length  $2m$  and speed  $m$  soliton breaks up into two length  $m$  and speed  $m$  solitons.

$\rightarrow$  Look at Ex 7 to explore more properties.



## Ex 10

$$\begin{aligned}
 1. \int_{-\infty}^{+\infty} dk e^{-A(k-\bar{k})^2 + iBk} &= e^{iB\bar{k}} \int_{-\infty}^{+\infty} dk e^{-A(k-\bar{k})^2 + iB(k-\bar{k})} = e^{iB\bar{k}} \int_{-\infty}^{+\infty} dh e^{-Ah^2 + iBh} \\
 &= e^{iB\bar{k}} \int_{-\infty}^{+\infty} dk e^{-A(h-i\frac{B}{2A})^2} \cdot e^{-\frac{B^2}{4A}} = \sqrt{\frac{\pi}{A}} e^{iB\bar{k}} e^{-B^2/4A}
 \end{aligned}$$

$$\begin{aligned}
 2. z(x,t) &= \int_{-\infty}^{+\infty} dk e^{-a^2(k-\bar{k})^2 + i[kx - \omega(k)t]} \\
 &\approx e^{i[\bar{k}x - \omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk e^{-a^2(k-\bar{k})^2 + i(k-\bar{k})x - i\omega'(\bar{k})(k-\bar{k})t - \frac{i}{2}\omega''(\bar{k})(k-\bar{k})^2 t} \\
 &= e^{i[\bar{k}x - \omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk e^{-[a^2 + \frac{i}{2}\omega''(\bar{k})t](k-\bar{k})^2 + i[x - \omega'(\bar{k})t](k-\bar{k})} \\
 &= e^{i[\bar{k}x - \omega(\bar{k})t]} \sqrt{\frac{\pi}{a^2 + \frac{i}{2}\omega''(\bar{k})t}} e^{-\frac{[x - \omega'(\bar{k})t]^2}{4[a^2 + \frac{i}{2}\omega''(\bar{k})t]}}
 \end{aligned}$$

3. The envelope, which is obtained by looking at the absolute value (thus neglecting the oscillatory part), is (recall that  $a \in \mathbb{R}$ )

$$|z| = \frac{\sqrt{2\pi}}{(4a^4 + \omega''(\bar{k})^2 t^2)^{1/4}} e^{-\frac{a^2}{4a^4 + \omega''(\bar{k})^2 t^2} \cdot [x - \omega'(\bar{k})t]^2}$$

Profile centred at  $x = \omega'(\bar{k})t$   
with width<sup>2</sup>  $\sim 4a^2 + \frac{\omega''(\bar{k})^2}{a^2} t^2$

By the way, this means that the amplitude of this wave decreases with time.

**Ex 11**

Sub in plane wave  $u(x,t) = e^{i(kx - \omega t)}$ :

(a)  $u_t + u_x + \alpha u_{xxx} = 0$

$\leadsto -i\omega + ik - i\alpha k^3 = 0 \Rightarrow \omega(k) = k - \alpha k^3$

Dispersion relation

Phase velocity  $c(k) = \frac{\omega(k)}{k} = 1 - \alpha k^2$

Group velocity  $c_g(k) = \omega'(k) = 1 - 3\alpha k^2$

(b)  $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}$

$\leadsto -\omega^2 + \alpha^2 k^2 = \beta^2 k^2 \omega^2 \Rightarrow \omega(k) = \pm \frac{\alpha k}{\sqrt{1 + \beta^2 k^2}}$

(We might be interested only in the + sign physically, but let's not restrict here)

Phase velocity  $c(k) = \frac{\omega(k)}{k} = \pm \frac{\alpha}{\sqrt{1 + \beta^2 k^2}}$

Group velocity  $c_g(k) = \omega'(k) = \pm \alpha \left[ \frac{1}{(1 + \beta^2 k^2)^{3/2}} - \frac{k}{\cancel{2} (1 + \beta^2 k^2)^{3/2}} \right] = \pm \frac{\alpha}{(1 + \beta^2 k^2)^{3/2}}$

**Ex 12**

$u_t + u_x + u_{xxx} + \underbrace{u_{x\dots x}}_n = 0$

Sub in plane wave  $u(x,t) = e^{i(kx - \omega t)}$ :

$-i\omega + ik - ik^3 + i^n k^n = 0$

$\omega(k) = k - k^3 + i^{n-1} k^n$

$\Rightarrow$  Plane wave  $u(x,t) = e^{ik[x - (1 - k^2)t]} e^{-i^n k^n t}$

Dissipation ( $\omega$  complex) if  $n \in 2\mathbb{Z}$  ( $n$  even).

Physical dissipation (amplitude decays) if  $n \in 4\mathbb{Z}$  ( $n$  is a multiple of 4).

Let  $n = 4m$ .  $u(x,t) = e^{ik[x - (1 - k^2)t]} e^{-k^{4m} t}$ .  $\rightarrow$  Exponential decay

**Ex 13.1**

$$\text{mKdV: } u_t + 6u^2u_x + u_{xxx} = 0$$

$$u, u_x, u_{xx} \xrightarrow{|x| \rightarrow \infty} 0 \quad \forall t$$

Sub in travelling wave  $u(x,t) = f(x-vt)$  :

$$-vf' + 6f^2f' + f''' = 0 \quad \text{for } f(\xi), \xi = x-vt$$

Integrate:  $-vf + 2f^3 + f'' = A$  ← integration constant

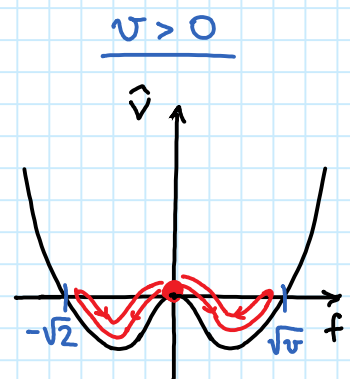
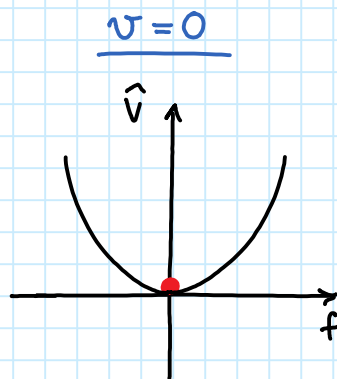
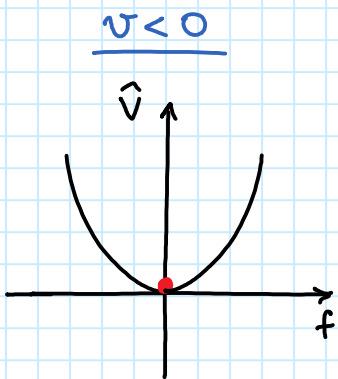
B.c.  $f, f', f'' \xrightarrow{|\xi| \rightarrow \infty} 0 \Rightarrow A = 0$ .

Multiply by  $f'$  and integrate:  $-\frac{v}{2}f^2 + \frac{1}{2}f^4 + \frac{1}{2}(f')^2 = B$

B.c.  $\Rightarrow B = 0$ .

So  $\frac{1}{2}(f')^2 + \frac{1}{2}f^2(f^2 - v) = 0$

$$\underbrace{\frac{1}{2}f^2(f^2 - v)}_{\hat{V}(f)} = 0$$



The b.c. tell us that we start at  $f=0$ , therefore a nontrivial travelling wave solution will only exist for  $v > 0$ , corresponding to rolling down and then up to  $\pm\sqrt{v}$ , and then back to  $f=0$ . (If  $v \leq 0$ , then  $f(\xi) \equiv 0$ .)

Therefore we anticipate that for  $v > 0$  there will be two travelling wave solutions. Let's find them analytically.

I'll focus on the solution with  $0 \leq f \leq \sqrt{v}$ , since the other solution can be found by sending  $f \rightarrow -f$ .

$$f' = \pm f \sqrt{v - f^2} \quad \Rightarrow \quad \int \frac{df}{f \sqrt{v - f^2}} = \pm \xi$$

Using the indefinite integral  $\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{arcsech}(x)$ , we obtain

$$\pm \xi = -\frac{1}{\sqrt{v}} \operatorname{arcsech}\left(\frac{f}{\sqrt{v}}\right) \pm x_0 \quad \text{integration constant}$$

$$\Rightarrow \text{Invert} \quad f = \sqrt{v} \operatorname{sech}\left[\mp \sqrt{v}(\xi - x_0)\right] = \sqrt{v} \operatorname{sech}\left[\sqrt{v}(\xi - x_0)\right]$$

Therefore the two travelling wave solutions are

$$u(x,t) = \pm \sqrt{v} \operatorname{sech}\left[\sqrt{v}(x - x_0 - vt)\right].$$

### Ex 13.3

" $\phi^4$ ":  $u_{tt} - u_{xx} + 2u(u^2 - 1) = 0$

$$u_t, u_x, u \rightarrow 1 \xrightarrow{x \rightarrow -\infty} 0$$

$$u_t, u_x, u \rightarrow -1 \xrightarrow{x \rightarrow +\infty} 0$$

Sub in travelling wave  $u(x,t) = f(x - vt) \equiv f(\xi)$ :

$$(1 - v^2)f'' = 2f(f^2 - 1) \iff f'' = 2\gamma^2 f(f^2 - 1), \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2}}$$

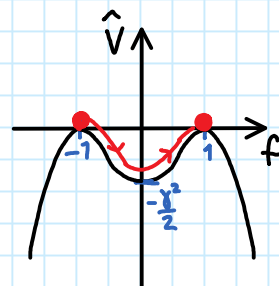
Multiply by  $f'$  and integrate:  $\frac{1}{2}(f')^2 = \gamma^2 \left(\frac{f^4}{2} - f^2\right) + A$

$$\text{B.C.} \Rightarrow 0 = \gamma^2 \left(\frac{1}{2} - 1\right) + A \Rightarrow A = \frac{\gamma^2}{2}$$

$$\text{So } \frac{1}{2}(f')^2 - \frac{1}{2}\gamma^2(f^2 - 1)^2 = 0$$

$$= \hat{V}(f)$$

$$\Rightarrow f' = \pm \gamma(f^2 - 1)$$



According to the b.c.'s, we look for a solution where  $f$  grows from  $-1$  to  $+1$ . This selects

$$f' = \gamma(1 - f^2) \implies \gamma \xi = \int \frac{df}{1 - f^2} = \operatorname{arctanh}(f) + \gamma x_0 \quad \text{integration constant}$$

Therefore the travelling wave solution is

$$u(x,t) = \tanh\left[\gamma(x - x_0 - vt)\right].$$

**Ex 13.5**

Burgers:  $u_t + uu_x - u_{xx} = 0$

$$u \rightarrow u_0, \quad u_x \rightarrow 0 \quad \text{as } x \rightarrow -\infty$$

$$u \rightarrow u_1, \quad u_x \rightarrow 0 \quad \text{as } x \rightarrow +\infty$$

$$(u_0 > u_1 > 0)$$

Sub in travelling wave  $u(x,t) = f(x-vt) \equiv f(\xi)$ :

$$-vf' + ff' - f'' = 0$$

$$\int d\xi: \quad -vf + \frac{1}{2}f^2 - f' = A \quad (*)$$

Impose B  $f \xrightarrow[\xi \rightarrow -\infty]{} u_0, \quad f \xrightarrow[\xi \rightarrow +\infty]{} u_1, \quad f' \xrightarrow[\xi \rightarrow \pm\infty]{} 0$ :

$$A = \frac{1}{2}u_0(u_0 - 2v) = \frac{1}{2}u_1(u_1 - 2v)$$

$$\Rightarrow \cancel{u_0^2} - \cancel{u_1^2} = 2v(u_0 - u_1) \Rightarrow v = \frac{1}{2}(u_0 + u_1)$$

$$\Rightarrow A = -\frac{1}{2}u_0u_1$$

Integrate (\*) by separation of variables:

$$\int \frac{df}{f^2 - (u_0 + u_1)f + u_0u_1} = 2 \int d\xi$$

$$\int \frac{1}{(f-u_0)(f-u_1)} = 2(\xi - x_0)$$

$$\frac{1}{u_0 - u_1} \int df \left( \frac{1}{f-u_0} - \frac{1}{f-u_1} \right) = 2(\xi - x_0)$$

$$\frac{1}{u_0 - u_1} \ln \left| \frac{f-u_0}{f-u_1} \right| = 2(\xi - x_0)$$

Need  $u_1 < f < u_0$ , so

$$\frac{u_0 - f}{f - u_1} = \exp \left[ \frac{2}{u_0 - u_1} (\xi - x_0) \right]$$

$$f(\xi) = \frac{u_0 + u_1 \exp \left[ \frac{2}{u_0 - u_1} (\xi - x_0) \right]}{1 + \exp \left[ \frac{2}{u_0 - u_1} (\xi - x_0) \right]}$$

$$\Rightarrow u(x,t) = \frac{u_0 + u_1 \exp \left[ \frac{2}{u_0 - u_1} \left( x - x_0 - \frac{1}{2}(u_0 + u_1)t \right) \right]}{1 + \exp \left[ \frac{2}{u_0 - u_1} \left( x - x_0 - \frac{1}{2}(u_0 + u_1)t \right) \right]}$$

CHECK:

$$\xrightarrow{x \rightarrow -\infty} u_0$$

$$\xrightarrow{x \rightarrow +\infty} u_1$$

## Ex 15

$$1. \quad F_n(\{\theta_m\}) = \underbrace{-a \sin \theta_n}_{\equiv F_n^{\text{grav}} = -\frac{\partial V^{\text{grav}}}{\partial \theta_n}} + \underbrace{\frac{1}{a} (\theta_{n+1} - \theta_n) + \frac{1}{a} (\theta_{n-1} - \theta_n)}_{\equiv F_n^{\text{twist}} = -\frac{\partial V^{\text{twist}}}{\partial \theta_n}} \equiv -\frac{\partial}{\partial \theta_n} V(\{\theta_m\})$$

This can be integrated to

$$V^{\text{grav}}(\{\theta_m\}) = a \sum_{m \in \mathbb{Z}} (1 - \cos \theta_m)$$

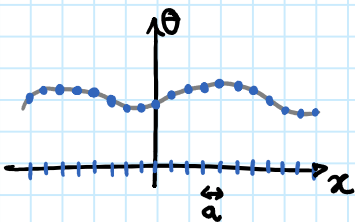
$$V^{\text{twist}}(\{\theta_m\}) = \frac{1}{2a} \sum_{m \in \mathbb{Z}} (\theta_{m+1} - \theta_m)^2$$

CHECK that partial derivatives reproduce the above forces!

where I've chosen the integration constants so that  $V^{\text{grav}} = 0$  when all pendula point down and  $V^{\text{twist}} = 0$  when all pendula are aligned. So

$$V(\{\theta_m\}) = a \sum_{n \in \mathbb{Z}} (1 - \cos \theta_n) + \frac{1}{2a} \sum_{n \in \mathbb{Z}} (\theta_{n+1} - \theta_n)^2.$$

2. In the CONTINUUM LIMIT, the set  $\{\theta_n(t)\}$  approximates a continuous function  $\theta(x,t)$ , s.t.  $\theta(x=na, t) = \theta_n(t)$ :

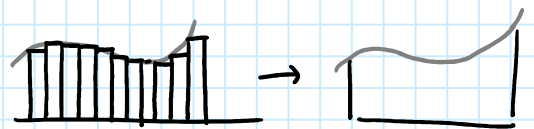


By the definition of the derivative as a limit

$$\frac{\theta_{n+1}(t) - \theta_n(t)}{a} \xrightarrow{a \rightarrow 0} \theta_x(x=na, t),$$

and by the definition of the integral as a limit of a sum

$$a \sum_{n \in \mathbb{Z}} f(\theta_n) \rightarrow \int_{-\infty}^{+\infty} dx f(\theta(x,t))$$



In our case

$$V = a \sum_{n \in \mathbb{Z}} \left[ (1 - \cos \theta_n) + \frac{1}{2} \left( \frac{\theta_{n+1} - \theta_n}{a} \right)^2 \right] \xrightarrow{a \rightarrow 0} \int_{-\infty}^{+\infty} dx \left[ (1 - \cos \theta(x,t)) + \frac{1}{2} \theta_x(x,t)^2 \right]$$

Similarly

$$T = \frac{a}{2} \sum_n \overset{\frac{d}{dt}}{\dot{\theta}_n(t)^2} \xrightarrow{a \rightarrow 0} \int_{-\infty}^{+\infty} dx \frac{1}{2} \theta_t(x,t)^2.$$

## Ex 16

$$1. \quad E = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] \quad (\lambda, a > 0)$$

$E < +\infty \Rightarrow u_t, u_x, u^2 - a^2 \rightarrow 0$  as  $x \rightarrow \pm\infty$ .  
therefore  $u \rightarrow a$  or  $-a$ .

2. There are 4 options according to the asymptotic values of  $u$  as  $x \rightarrow \pm\infty$ :

B.C. $u(-\infty, t)$	B.C. $u(+\infty, t)$	Travelling wave sol'n $u(x, t)$
$-a$	$-a$	$-a$
$+a$	$+a$	$+a$
$-a$	$+a$	$+a \cdot \tanh[a\sqrt{\lambda}\gamma(x-x_0-vt)]$
$+a$	$-a$	$-a \cdot \tanh[a\sqrt{\lambda}\gamma(x-x_0-vt)]$

where the nontrivial travelling wave solutions

$$u(x, t) = \pm a \tanh[a\sqrt{\lambda}\gamma(x-x_0-vt)]$$

are obtained as in exercise 13.3 (the only difference being  $a, \lambda \neq 1$ ).

The constant solutions have 0 energy, whereas for the nontrivial solutions

$$u_t^2 = a^4 \lambda \gamma^2 v^2 \operatorname{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)]$$

$$u_x^2 = a^4 \lambda \gamma^2 \cdot \operatorname{sech}^4 \left[ \begin{array}{c} \text{''} \\ \text{''} \end{array} \right]$$

$$\lambda(u^2 - a^2)^2 = \lambda a^4 \cdot \operatorname{sech}^4 \left[ \begin{array}{c} \text{''} \\ \text{''} \end{array} \right]$$

using  $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$   
and  $\tanh^2 x - 1 = -\operatorname{sech}^2 x$

$$\begin{aligned} \Rightarrow E &= \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[ u_t^2 + u_x^2 + \lambda(u^2 - a^2)^2 \right] = \frac{\lambda a^4}{2} (1 + \gamma^2(1 + v^2)) \int_{-\infty}^{\infty} dx \operatorname{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)] \\ &= \frac{\sqrt{\lambda} a^3}{2\gamma} \cdot \underbrace{\frac{1-v^2+1+v^2}{1-v^2}}_{=2\gamma^2} \cdot \underbrace{\int_{-\infty}^{\infty} dy \operatorname{sech}^4(y)}_{=\frac{8}{6}=\frac{4}{3}} = \frac{4}{3} a^3 \sqrt{\lambda} \cdot \gamma, \quad \text{with } \gamma = \frac{1}{\sqrt{1-v^2}}. \end{aligned}$$

Static solutions ( $v=0 \Leftrightarrow \gamma=1$ ) have the lowest energy:  $M = E|_{v=0} = \frac{4}{3} a^3 \sqrt{\lambda}$ .

$$\begin{aligned}
3. \quad E &= \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] = \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[ u_t^2 + (u_x \mp \sqrt{\lambda} (u^2 - a^2))^2 \pm 2\sqrt{\lambda} u_x (u^2 - a^2) \right] \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[ u_t^2 + (u_x \mp \sqrt{\lambda} (u^2 - a^2))^2 \right] \pm \sqrt{\lambda} \left[ \frac{u^3}{3} - a^2 u \right]_{-\infty}^{+\infty} \\
&\geq \pm \sqrt{\lambda} \left[ \frac{u^3}{3} - a^2 u \right]_{-\infty}^{+\infty}
\end{aligned}$$

For the "kink"  $\lim_{x \rightarrow \pm\infty} u = \pm a$ , therefore

$$E \geq \pm \sqrt{\lambda} \cdot 2 \cdot \left( \frac{a^3}{3} - a^2 \cdot a \right) = \mp \frac{4}{3} \sqrt{\lambda} a^3 \Rightarrow E \geq \frac{4}{3} \sqrt{\lambda} a^3.$$

$$E = \frac{4}{3} \sqrt{\lambda} a^3 \quad \text{iff} \quad u_t = 0 \quad \text{and} \quad u_x + \sqrt{\lambda} (u^2 - a^2) = 0.$$

Integrating,  $\int \frac{du}{a^2 - u^2} = \sqrt{\lambda} \int dx \Rightarrow \frac{1}{a} \operatorname{arctanh} \frac{u}{a} = \sqrt{\lambda} (x - x_0)$

$$\Rightarrow u(x, t) = a \tanh [a\sqrt{\lambda} (x - x_0)],$$

which is indeed the static kink found in part 2.



## Ex 18

$$x \in [-\pi/2, \pi/2] \quad E = \int_{-\pi/2}^{\pi/2} dx \frac{1}{2} (u_t^2 + u_x^2 + 1 - u^2)$$

$u(x,t)$  satisfies  $|u(x,t)| \leq 1$  and BC  $|u(\pm \frac{\pi}{2}, t)| = 1$ .

$$E \geq \frac{1}{2} \int_{-\pi/2}^{\pi/2} dx (u_x \pm \sqrt{1-u^2})^2 \mp \int_{-\pi/2}^{\pi/2} dx u_x \sqrt{1-u^2} \quad (= \text{iff } u_t = 0)$$

$$\geq \mp \int_{u(-\frac{\pi}{2}, t)}^{u(\frac{\pi}{2}, t)} du \sqrt{1-u^2} \quad (= \text{iff } u_x = \mp \sqrt{1-u^2})$$

$$\geq \mp \frac{1}{2} \left[ u \sqrt{1-u^2} + \arcsin u \right]_{u(-\frac{\pi}{2}, t)}^{u(\frac{\pi}{2}, t)}$$

"kink":  $u(-\frac{\pi}{2}, t) = -1$ ,  $u(+\frac{\pi}{2}, t) = +1$

$$\left[ \dots \right]_{-1}^{+1} = \arcsin(1) - \arcsin(-1) = \pi.$$

Hence  $E \geq \frac{\pi}{2}$ , picking the lower sign.

The bound is saturated if  $u_t = 0$  and  $u_x = \sqrt{1-u^2}$ . Integrating,

$$\int \frac{du}{\sqrt{1-u^2}} = \int dx \Rightarrow \arcsin u = x - x_0$$

Impose BC: 
$$\begin{cases} \arcsin(-1) = -\frac{\pi}{2} - x_0 \\ \arcsin(1) = +\frac{\pi}{2} - x_0 \end{cases} \Rightarrow x_0 = 0$$

$\Rightarrow u(x) = \sin x$  saturates the lower bound.

CHECK: 
$$E = \frac{1}{2} \int_{-\pi/2}^{\pi/2} dx \left[ u_t^2 + u_x^2 + 1 - u^2 \right] = \frac{1}{2} \int_{-\pi/2}^{\pi/2} dx \left[ 0 + \cos^2 x + 1 - \sin^2 x \right] = \int_{-\pi/2}^{\pi/2} dx \frac{1 + \cos 2x}{2} = \frac{\pi}{2}.$$



## Ex 20.1

Sine-Gordon kink of velocity  $v$ :  $u(x,t) = 4 \arctan \left( e^{\gamma(x-x_0-vt)} \right)$ .

Up to normalization, the TOPOLOGICAL CHARGE is

$$Q_0 \propto \frac{1}{2\pi} [u]_{-\infty}^{+\infty} = \frac{2\pi - 0}{2\pi} = 1.$$

(Any normalization is fine. I just wanted to see  $Q_0 \propto [u]_{-\infty}^{+\infty}$ .)

The ENERGY is

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + (1 - \cos u) \right] = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} (1+v^2) u_x^2 + 2 \sin^2 \frac{u}{2} \right] \\ &= \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} (1+v^2) \cdot \gamma^2 4 \operatorname{sech}^2(\gamma(x-x_0-vt)) + 2 \sin^2(2 \arctan(e^{\gamma(x-x_0-vt)})) \right] \\ &= \frac{1}{\gamma} \int_{-\infty}^{+\infty} dy \left[ 2 \frac{1+v^2}{1-v^2} \operatorname{sech}^2(y) + 8 \sin^2(\arctan(e^y)) \cos^2(\arctan(e^y)) \right] \end{aligned}$$

Let  $\theta \equiv \arctan(e^y)$ . Then the last term is:

$$8 \sin^2 \theta \cdot \cos^2 \theta = 8 \tan^2 \theta \cdot \cos^4 \theta = 8 \frac{\tan^2 \theta}{(\tan^2 \theta + 1)^2} = 2 \left( \frac{2}{\tan \theta + \frac{1}{\tan \theta}} \right)^2 = 2 \cdot \left( \frac{2}{e^y + e^{-y}} \right)^2 = 2 \operatorname{sech}^2(y).$$

$$\Rightarrow E = \frac{1}{\gamma} \int_{-\infty}^{+\infty} dy 2 \left( \frac{1+v^2}{1-v^2} + 1 \right) \operatorname{sech}^2(y) = 4\gamma \int_{-\infty}^{+\infty} dy \operatorname{sech}^2(y) = 8\gamma = \frac{8}{\sqrt{1-v^2}}.$$

The MOMENTUM is

$$P = - \int_{-\infty}^{+\infty} dx u_t u_x = v \int_{-\infty}^{+\infty} dx u_x^2 = 4v \gamma^2 \int_{-\infty}^{+\infty} dx \operatorname{sech}^2(\gamma(x-x_0-vt)) = 4\gamma v \int_{-\infty}^{+\infty} dy \operatorname{sech}^2(y) = 8\gamma v.$$

All these 3 charges are conserved (t-independent).

In terms of the mass  $M = E|_{v=0} = 8$ ,

$$E = M\gamma = \frac{M}{\sqrt{1-v^2}} = M + \frac{1}{2} Mv^2 + O(Mv^4)$$

$$P = M\gamma v = Mv + O(Mv^3).$$

## Ex 23

1. For  $\rho_1$  and  $\rho_2$ , see the lecture notes. For  $\rho_*$ :

$$(\rho_* - 3tu^2)_t = \rho_{*t} - 3u^2 - 6tuu_t \stackrel{\text{KdV}}{=} -x(6uu_x + u_{xxx}) - 3u^2 + 6tu(6uu_x + u_{xxx})$$

Term by term:

$$-6xuu_x = -(3xu^2)_x + 3u^2$$

$$-x u_{xxx} = -(xu_{xx})_x + u_{xx} = (-xu_{xx} + u_x)_x - 3u^2$$

$$36tu^2u_x = (12tu^3)_x$$

$$6tuu_{xxx} = (6tuu_{xx})_x - 6tu_x u_{xx} = (6tuu_{xx} - 3tu_x^2)_x$$

$$\Rightarrow (\rho_*)_t = (\rho_* - 3tu^2)_t = (-3xu^2 - xu_{xx} + u_x + 12tu^3 + 6tuu_{xx} - 3tu_x^2)_x \equiv -(j_*)_x,$$

and assuming that  $u$  and its  $x$ -derivatives tend to 0 fast enough as  $x \rightarrow \pm\infty$ ,

$$Q_* = \int_{-\infty}^{\infty} dx (\rho_* - 3tu^2)$$

is a conserved charge like  $Q_1$  and  $Q_2$ .

(More precisely: the usual b.c.'s imply that  $u, u_x, u_{xx}, \dots \rightarrow 0$  as  $|x| \rightarrow \infty \forall t$ .  
 In order to have a conserved charge  $Q_*$ , however, we need  $\lim_{x \rightarrow +\infty} j_* = \lim_{x \rightarrow -\infty} j_*$  and  $Q_*$  finite.  
 For the latter, we need  $u \rightarrow 0$  faster than  $x^{-2}$ , unless there are cancellations.)

2.  $u_{\mu, x_0}(x, t) = 2\mu^2 \operatorname{sech}^2[\mu(x - x_0 - 4\mu^2 t)]$

(I will use  $\int_{-\infty}^{+\infty} dx \operatorname{sech}^{2n} x = \frac{2^{2n-1} (n-1)!^2}{(2n-1)!}$ .)

$$Q_1 = \int_{-\infty}^{+\infty} dx u_{\mu, x_0}(x, t) = 2\mu^2 \int_{-\infty}^{+\infty} dx \operatorname{sech}^2(\mu x) = 2\mu \int_{-\infty}^{+\infty} dy \operatorname{sech}^2 y = 2\mu \cdot 2 = 4\mu$$

$$Q_2 = \int_{-\infty}^{+\infty} dx u_{\mu, x_0}^2(x, t) = 4\mu^4 \int_{-\infty}^{+\infty} dx \operatorname{sech}^4(\mu x) = 4\mu^3 \int_{-\infty}^{+\infty} dy \operatorname{sech}^4 y = 4\mu^3 \cdot \frac{2^3}{3!} = \frac{16}{3} \mu^3$$

$$Q_* = \int_{-\infty}^{+\infty} dx \left[ x u_{\mu, x_0} - 3t u_{\mu, x_0}^2 \right] = 2\mu^2 \int_{-\infty}^{+\infty} dx x \cdot \operatorname{sech}^2[\mu(x-x_0-4\mu^2 t)] - 3t Q_2$$

NOTE: integrable because  $\operatorname{sech}^2 x \rightarrow 0$  exponentially as  $|x| \rightarrow \infty$

$$= -3t Q_2 + 2\mu^2 \left\{ \int_{-\infty}^{+\infty} dx (x-x_0-4\mu^2 t) \operatorname{sech}^2[\mu(x-x_0-4\mu^2 t)] + (x_0+4\mu^2 t) \int_{-\infty}^{+\infty} dx \operatorname{sech}^2[\mu(x-x_0-4\mu^2 t)] \right\}$$

$$= (x_0+4\mu^2 t) Q_1 - 3t Q_2$$

0 since it's the integral of an odd (and integrable) function

$$= (x_0+4\mu^2 t) \cdot 4\mu - 3t \cdot \frac{16}{3} \mu^2 = 4\mu x_0,$$

which indeed does not depend on time.

SUMMARY: for the 1-soliton  $u_{\mu, x_0}(x, t) = 2\mu^2 \operatorname{sech}^2[\mu(x-x_0-4\mu^2 t)]$ ,

$$Q_1 = 4\mu, \quad Q_2 = \frac{16}{3} \mu^3 \quad \text{and} \quad Q_* = 4\mu x_0 \quad \text{are conserved charges.}$$

3. Since  $Q_1$  &  $Q_2$  are conserved, they must have the same values for the initial condition  $u(x, t=0)$  and for the late time solution  $u(x, t \rightarrow +\infty)$  which consists of two well-separated solitons.

$$\underline{t=0}: \quad u(x, 0) = 6 \operatorname{sech}^2 x$$

$$Q_1 = \int_{-\infty}^{+\infty} dx u(x, 0) = 6 \int_{-\infty}^{+\infty} dx \operatorname{sech}^2 x = 12$$

$$Q_2 = \int_{-\infty}^{+\infty} dx u^2(x, 0) = 36 \int_{-\infty}^{+\infty} dx \operatorname{sech}^4 x = 36 \cdot \frac{4}{3} = 48.$$

$$\underline{t \rightarrow +\infty}: \quad u(x, t) \approx u_{\mu_1, x_1}(x, t) + u_{\mu_2, x_2}(x, t) \quad (\text{i.e. } \lim_{t \rightarrow +\infty} (\text{LHS} - \text{RHS}) = 0)$$

↑  
Sum of two well-separated solitons. Charges add up:

$$Q_1 = 4\mu_1 + 4\mu_2 = 4(\mu_1 + \mu_2)$$

$$Q_2 = \frac{16}{3} (\mu_1^3 + \mu_2^3).$$

Equating the (conserved) charges of the solution at  $t=0$  and  $t \rightarrow +\infty$  we find:

$$\begin{cases} 4(\mu_1 + \mu_2) = 12 \\ \frac{16}{3}(\mu_1^3 + \mu_2^3) = 48 \end{cases} \Rightarrow \begin{cases} \mu_1 + \mu_2 = 3 \\ (\cancel{\mu_1 + \mu_2})(\mu_1^2 - \mu_1\mu_2 + \mu_2^2) = \frac{3}{\cancel{9}} \end{cases} \Rightarrow \begin{cases} \mu_1 + \mu_2 = 3 \\ (\mu_1 + \mu_2)^2 - 3\mu_1\mu_2 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} \mu_1 + \mu_2 = 3 \\ \mu_1\mu_2 = 2 \end{cases} \Rightarrow (\mu_1, \mu_2) = (1, 2) \text{ or } (2, 1).$$

Therefore the velocities of the two solitons which appear at late times are  $v_i = 4\mu_i^2 = 4, 16$ , and their heights are  $2\mu_i^2 = 2, 8$ , in agreement with the animations seen in the 1<sup>st</sup> week.

↖ file N=2

4. Same logic as above, now for

$$u(x, t) \underset{t \rightarrow -\infty}{\approx} u_{\mu_1, x_1}(x, t) + u_{\mu_2, x_2}(x, t)$$

$$u(x, t) \underset{t \rightarrow +\infty}{\approx} u_{\mu_1, y_1}(x, t) + u_{\mu_2, y_2}(x, t)$$

and the conserved charge  $Q_*$ :

$$Q_* \Big|_{t \rightarrow -\infty} = 4(\mu_1 x_1 + \mu_2 x_2)$$

$$\stackrel{||}{=} Q_* \Big|_{t \rightarrow +\infty} = 4(\mu_1 y_1 + \mu_2 y_2)$$

$$\Rightarrow \mu_1(y_1 - x_1) = -\mu_2(y_2 - x_2) \Rightarrow y_2 - x_2 = -\frac{\mu_1}{\mu_2}(y_1 - x_1).$$