SOLUTIONS TO SELECTED EXERCISES

Ex 1

Sub $u(x,t) = \dfrac{2}{\cosh^2(x-vt)}$ in KdV eqn $u_t + 6uu_x + u_{xxx} = 0$:

$$u_t = \frac{4v}{\cosh^3(x-vt)} \cdot \sinh(x-vt)$$

$$u_x = \frac{-4}{\cosh^3(x-vt)} \sinh(x-vt)$$

$$u_{xx} = \frac{12}{\cosh^4(x-vt)} \sinh^2(x-vt) - \frac{4}{\cosh^2(x-vt)} = \frac{8}{\cosh^2(x-vt)} - \frac{12}{\cosh^4(x-vt)}$$

$\cosh^2 x - \sinh^2 x = 1$

$$u_{xxx} = \left[\frac{-16}{\cosh^3(x-vt)} + \frac{48}{\cosh^5(x-vt)}\right] \sinh(x-vt)$$

So KdV $\Rightarrow$ $$0 = \frac{4\sinh(x-vt)}{\cosh^3(x-vt)} \cdot \left[v - \cancel{\frac{48}{\cosh^2(x-vt)}} - 4 + \cancel{\frac{48}{\cosh^2(x-vt)}} \right].$$

Therefore $u = \dfrac{2}{\cosh^2(x-vt)}$ is a solution of KdV if (and only if) $\underline{\underline{v=4}}$.

Ex 2

1. Let $u(x,t) = A\,g(X,T)$, where $X = Bx$ and $T = Ct$.

Chain rule $\Rightarrow$ $u_t(x,t) = AC\,g_T(X,T)$, $u_x(x,t) = AB\,g_X(X,T)$, $u_{xxx}(x,t) = AB^3 g_{XXX}(X,T)$

$$\Rightarrow \quad u_t + 6uu_x + u_{xxx} = AC\,g_T(X,T) + 6A^2B\,g(X,T)\,g_X(X,T) + AB^3 g_{XXX}(X,T)$$

$$= AC \cdot \left[g_T(X,T) + \frac{AB}{C} \cdot 6g(X,T)\,g_X(X,T) + \frac{B^3}{C} g_{XXX}(X,T) \right].$$

If $g(X,T)$ solves KdV (in its variables X and T !), then $u(x,t)$ solves KdV (in its variables x and t) provided that $\frac{AB}{C} = \frac{B^3}{C} = 1$, because in that case

$$u_t + 6uu_x + u_{xxx} = AC\left[g_T(X,T) + 6g(X,T)g_X(X,T) + g_{XXX}(X,T)\right] = 0.$$

$\frac{AB}{C} = \frac{B^3}{C} = 1$ KdV for $g(X,T)$

$$\frac{AB}{C} = \frac{B^3}{C} = 1 \quad \Longleftrightarrow \quad \underline{A = B^2, \ C = B^3}.$$

2. $u(x,t) = g(x,t) \equiv \dfrac{2}{\cosh^2(x-4t)}$ 1-soliton solution of KdV

$$\Rightarrow u(x,t) = B^2 g(Bx, B^3 t) = \frac{2B^2}{\cosh^2[B(x-4B^2t)]} \equiv u_B(x,t)$$

label, not derivative!

1-parameter family of 1-soliton sol'ns (with parameter B)

3. (Height of u_B) $= 2B^2$
(Velocity of u_B) $= 4B^2$ $\Longrightarrow$ Velocity $= 2$ Height.

The "width" is a measure of how the lump is concentrated in space (x). Since the dependence of u_B on x is only through Bx, we can say that (Width of u_B) $\propto \frac{1}{B}$. The precise proportionality factor depends on the precise definition of width that you might choose, but regardless of that choice

(B>0 without loss of generality)

$$v \longrightarrow 4v = B^2 v \Longrightarrow \text{width} \longrightarrow \frac{1}{B}\,\text{width} = \frac{\text{width}}{2}.$$

Don't worry if you were confused by part 3 because I didn't define the width unambiguously. My aim was to make you think about the meaning of width (and see how many of you would realize that the answer was largely independent of the def'n).

I might run a few more social experiments like this during the year, but not in the exam, I promise. And feel free to ask me questions about the homework if you think you need more information than is given in the text.

SUMMARY of this exercise: For 1-soliton solutions of KdV

$$\text{Height} \propto \text{Velocity} \propto \text{Width}^{-2}.$$

$\boxed{\text{Ex 3}}$

Direct method: check using the chain rule.

- $v_t + 6\varepsilon v v_x + v_{xxx} = \frac{1}{\varepsilon} u_t + 6\varepsilon \frac{1}{\varepsilon^2} u u_x + \frac{1}{\varepsilon} u_{xxx} = \frac{1}{\varepsilon}(u_t + 6 u u_x + u_{xxx}) = 0.$

- $w_t + 6 w w_x + \varepsilon w_{xxx} = \varepsilon \underbrace{\frac{\partial T}{\partial t}}_{=\varepsilon} \frac{\partial u(x,T)}{\partial T}\Big|_{T=\varepsilon t} + 6\varepsilon^2 u(x,T) u_x(x,T)\Big|_{T=\varepsilon t}$

$$+ \varepsilon \cdot \varepsilon\, u_{xxx}(x,T)\Big|_{T=\varepsilon t} = \varepsilon^2 \left[u_T(x,T) + 6u(x,T)u_x(x,T) + u_{xxx}(x,T)\right]\Big|_{T=\varepsilon t} = 0.$$

Alternatively, suppose you didn't know the final eqn for v, w and wanted to derive it. Express u in terms of v (/w) and sub in KdV eqn:

- $u(x,t) = \varepsilon v(x,t)$: KdV $\Rightarrow$ $\varepsilon v_t + 6\varepsilon^2 v v_x + \varepsilon v_{xxx} = 0$
  $\Rightarrow$ $v_t + 6\varepsilon v v_x + v_{xxx} = 0.$

- $u(x,\varepsilon t) = \frac{1}{\varepsilon} w(x,t)$: KdV for $u(x,\varepsilon t)$ is

$$0 = \frac{\partial u(x,\varepsilon t)}{\partial(\varepsilon t)} + 6u(x,\varepsilon t)u_x(x,\varepsilon t) + u_{xxx}(x,\varepsilon t)$$

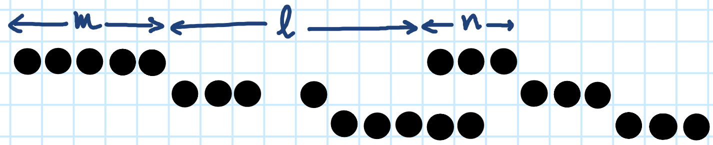
from $u(x,\varepsilon t) = \frac{1}{\varepsilon} w(x,t)$

from $\frac{\partial t}{\partial(\varepsilon t)}$

$$= \frac{1}{\varepsilon} \cdot \frac{1}{\varepsilon} w_t(x,t) + \frac{6}{\varepsilon^2} w(x,t) w_x(x,t) + \frac{1}{\varepsilon} w_{xxx}(x,t)$$

$$= \frac{1}{\varepsilon^2}\left[w_t + 6 w w_x + \varepsilon w_{xxx}\right]$$

Ex 4



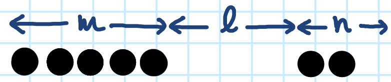
When the length m soliton is far enough to the left of the length n soliton (with $m>n$), we can evolve the system forward by one unit of time, to reduce the separation ℓ:

$$\ell \longrightarrow \ell - m + n .$$

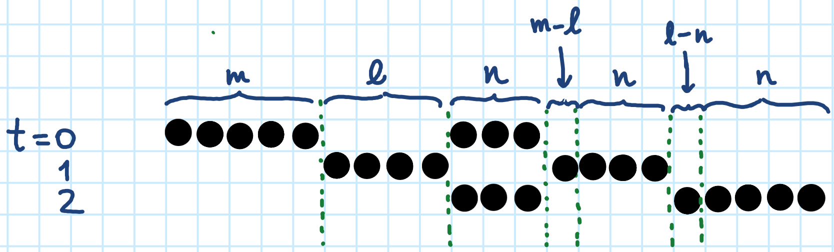
We can iterate this until the separation ℓ reaches the range

$$n \leq \ell < m,$$

and take this as our starting point:



Let's evolve this configuration forward and see what happens to the two solitons. To cover all cases, we'll have to evolve the system by $\Delta t = 2$.



Make sure you understand how this follows from the ball and box rule.

If you don't, ask me.

Unless $n=\ell$, at $t=1$ the system is in an intermediate configuration, still in the middle of the interaction. But already at $t=2$ the faster soliton has overtaken the slower, after which they keep travelling undisturbed to the right, with increasing separation.

To calculate the phase shifts of the two solitons, let's compare their positions at $t=2$ (after the interaction) with the positions they would have had at the same time had the other soliton not been there.

Compared to $t=0$, the length m soliton has moved to the right by

$$m+\ell+n+(m-\ell)+n = 2m+2n$$

boxes. It would have moved by $2m$ boxes in the absence of an interaction, so its phase shift is $+2n$.

Compared to $t=0$, the length n soliton has moved to the right by 0 boxes. It would have moved by $2n$ boxes had no interaction taken place, therefore its phase shift is $-2n$.

CONCLUSION :

The two solitons emerge from the interaction with the same lengths and velocities, but advanced/delayed by

$$|\text{Phase shift}| = 2\times(\text{Length of the slower soliton}).$$

- What can go "wrong" if $\ell < n$?

The starting point is in the middle of the interaction! See for instance $t=1$ in the picture above. This is just an intermediate configuration, which can involve different objects than the solitons appearing in the far past/future.

Ex 5

To make my life easier and aid memory, I will avoid colours and use instead "1" for the (blue) ball which is moved first and "2" for the (red) ball which is moved afterwards.

E.g.

1 1 2 ↕ 2 2 1   2   1 2

- Clearly:

  A row of n consecutive 1's is a speed n soliton

  " " " " " 2's " " " " "

because if a single colour/type of ball is present, it evolves as in the single colour ball and box model.

```
1 1 1                           2 2 2 2
      1 1 1                             2 2 2 2
            1 1 1                               2 2 2 2
```

- Next, consider a sequence/row where 1's are separated by 2's, e.g.

```
1 1 2 1 2 2 1
    2     2 1 1 1 1 2
      2     2         1 1 1 1 2
```

Since 1's move first, after one unit of time all 1's will be consecutive, so this is not a soliton. In order to have a soliton, all the 1's must be consecutive!

- Consider then a row of m 2's followed by n-m 1's:

$$\overbrace{1\,1\,1\,1}^{n-m}\,\overbrace{2\,2\,2}^{m}$$
$$\qquad\qquad\qquad 1\,1\,1\,1\,2\,2\,2$$

1's move first, translating by n boxes to the right, and 2's follow.

⟶ Speed n soliton.

This works for $m = 0, 1, 2, \dots, n$ ⇒ <u>n+1</u> <u>speed n</u> solitons.

- What if we have m 1's followed by n-m 2's?

$\overbrace{2222}^{n-m\,>}\overbrace{111}^{m}$
$\quad\;\;222\,1112$

$\overbrace{222}^{n-m=}\overbrace{111}^{m}$
$\quad\;\;222111$

$\overbrace{22}^{n-m<}\overbrace{111}^{m}$
$\quad\;\;22\;\;111$

1's move first by m units. 2's follow:

1) $n-m>m$: at least one 2 overtakes the row of 1's.

2) $n-m=m$: final configuration = initial configuration.
$\rightarrow$ <u>Speed $m=\frac{n}{2}$ soliton</u> (if $n\in 2\mathbb{Z}$).

3) $n-m<m$: the row of 2's is left behind.

- Finally, what about

$$\overbrace{2222}^{a}\overbrace{111}^{b}\overbrace{22222}^{c} \quad ?$$

1's move first, leaving a gap between the two rows of 2's. The gap is (partially or completely) filled by some 2's from the left group, but anyway the 2's from the right group leave a gap of (at least) c empty boxes to the left of the row of 1's.

$$222\underbrace{\qquad\qquad}_{c}111222222 \qquad \rightarrow \text{Not a soliton!}$$

<u>SUMMARY</u>:

- <u>n is odd</u>: there are n+1 length n solitons, all of speed n.

$$\overbrace{11111}^{n-m}\overbrace{222}^{m} \qquad m=0,1,2,\dots,n.$$

- <u>n is even</u>: in addition to the above, there's an extra length n soliton of speed $\frac{n}{2}=m$.

$\overbrace{222}^{m}\overbrace{111}^{m}$
$\quad\;\;222111$

In fact, this latter soliton is better thought of as the union of two consecutive length $\frac{n}{2}$ and speed $\frac{n}{2}$ solitons.

Indeed, consider the scattering (→ Ex 7)

```
speed 3       speed 2
122           2211
   122            2211
       122            2211
           12            22112
             12            22   112
                12            22     112
                  12            22      112
                speed 2      speed 2    speed 3
```

Note that numbers (/colours) are reshuffled, but lengths and speeds are unchanged.
The length $2m$ and speed m soliton breaks up into two length m and speed m solitons.

→ Look at Ex 7 to explore more properties.

Ex 10

1. $$\int_{-\infty}^{+\infty} dk\, e^{-A(k-\bar{k})^2+iBk} = e^{iB\bar{k}} \int_{-\infty}^{+\infty} dk\, e^{-A(k-\bar{k})^2+iB(k-\bar{k})} \underset{k=\bar{k}+h}{=} e^{iB\bar{k}} \int_{-\infty}^{+\infty} dh\, e^{-Ah^2+iBh}$$

$$= e^{iB\bar{k}} \int_{-\infty}^{+\infty} dk\, e^{-A\left(h-i\frac{B}{2A}\right)^2} \cdot e^{-\frac{B^2}{4A}} = \sqrt{\frac{\pi}{A}}\, e^{iB\bar{k}}\, e^{-B^2/4A}$$

2. $$z(x,t) = \int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2+i[kx-\omega(k)t]}$$

$$\approx e^{i[\bar{k}x-\omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2+i(k-\bar{k})x-i\omega'(\bar{k})(k-\bar{k})t-\frac{i}{2}\omega''(\bar{k})(k-\bar{k})^2 t}$$

$$= e^{i[\bar{k}x-\omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk\, e^{-\left[a^2+\frac{i}{2}\omega''(\bar{k})t\right](k-\bar{k})^2+i\left[x-\omega'(\bar{k})t\right](k-\bar{k})}$$

$$= e^{i[\bar{k}x-\omega(\bar{k})t]} \sqrt{\frac{\pi}{a^2+\frac{i}{2}\omega''(\bar{k})t}}\; e^{-\frac{[x-\omega'(\bar{k})t]^2}{4\left[a^2+\frac{i}{2}\omega''(\bar{k})t\right]}}$$

3. The envelope, which is obtained by looking at the absolute value (thus neglecting the oscillatory part), is (recall that $a \in \mathbb{R}$)

$$|z| = \frac{\sqrt{2\pi}}{\left(4a^4+\omega''(\bar{k})^2t^2\right)^{\frac{1}{4}}}\, e^{-\frac{a^2}{4a^4+\omega''(\bar{k})^2t^2}\cdot[x-\omega'(\bar{k})t]^2}.$$

Profile centred at $x=\omega'(\bar{k})t$ with width$^2 \sim 4a^2+\frac{\omega''(\bar{k})^2}{a^2}t^2$

By the way, this means that the amplitude of this wave decreases with time.

Ex 11 Sub in plane wave $u(x,t) = e^{i(kx-\omega t)}$:

(a) $u_t + u_x + \alpha u_{xxx} = 0$

$\rightsquigarrow \; -i\omega + ik - i\alpha k^3 = 0 \quad \Rightarrow \quad \omega(k) = k - \alpha k^3$ Dispersion relation

Phase velocity $c(k) = \dfrac{\omega(k)}{k} = 1 - \alpha k^2$

Group velocity $c_g(k) = \omega'(k) = 1 - 3\alpha k^2$

(b) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}$

$\rightsquigarrow \; -\omega^2 + \alpha^2 k^2 = \beta^2 k^2 \omega^2 \quad \Rightarrow \quad \omega(k) = \pm \dfrac{\alpha k}{\sqrt{1+\beta^2 k^2}}$ (We might be interested only in the + sign physically, but let's not restrict here)

Phase velocity $c(k) = \dfrac{\omega(k)}{k} = \pm \dfrac{\alpha}{\sqrt{1+\beta^2 k^2}}$

Group velocity $c_g(k) = \omega'(k) = \pm \alpha \left[\dfrac{1}{(1+\beta^2k^2)^{1/2}} - \dfrac{k}{\cancel{2}} \dfrac{\cancel{2}\beta^2 k}{(1+\beta^2k^2)^{3/2}} \right] = \pm \dfrac{\alpha}{(1+\beta^2 k^2)^{3/2}}$

Ex 12 $u_t + u_x + u_{xxx} + \underbrace{u_{x\ldots x}}_{n} = 0$

Sub in plane wave $u(x,t) = e^{i(kx-\omega t)}$:

$-i\omega + ik - ik^3 + i^n k^n = 0$

$\omega(k) = k - k^3 + i^{n-1} k^n$

$\Rightarrow$ Plane wave $u(x,t) = e^{ik[x-(1-k^2)t]} \, e^{-i^n k^n t}$

Dissipation (ω complex) if $n \in 2\mathbb{Z}$ (n even).

Physical dissipation (amplitude decays) if $n \in 4\mathbb{Z}$ (n is a multiple of 4).

Let $n = 4m$. $u(x,t) = e^{ik[x-(1-k^2)t]} \, e^{-k^{4m} t}$. $\rightarrow$ Exponential decay

Ex 13.1

mKdV: $u_t + 6u^2 u_x + u_{xxx} = 0$

$u, u_x, u_{xx} \xrightarrow[|x|\to\infty]{} 0 \quad \forall t$

Sub in travelling wave $u(x,t) = f(x - vt)$:

$$-vf' + 6f^2 f' + f''' = 0 \qquad \text{for } f(\xi), \ \xi = x - vt$$

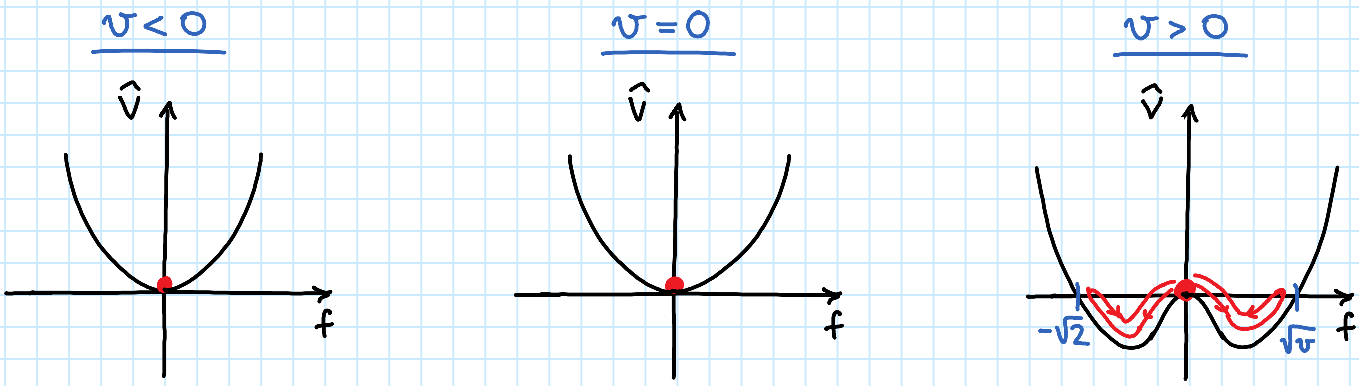
Integrate : $-vf + 2f^3 + f'' = A$ ← integration constant

B.c. $f, f', f'' \xrightarrow[|\xi|\to\infty]{} 0 \Rightarrow A = 0$.

Multiply by f' and integrate : $-\frac{v}{2} f^2 + \frac{1}{2} f^4 + \frac{1}{2}(f')^2 = B$

B.c. $\Longrightarrow B = 0$.

So
$$\frac{1}{2}(f')^2 + \underbrace{\frac{1}{2} f^2 (f^2 - v)}_{\equiv \hat{V}(f)} = 0 .$$



The b.c. tell us that we start at $f=0$, therefore a nontrivial travelling wave solution will only exist for $v>0$, corresponding to rolling down and then up to $\pm\sqrt{v}$, and then back to $f=0$. (If $v \le 0$, then $f(\xi) \equiv 0$.)

Therefore we anticipate that for $v>0$ there will be two travelling wave solutions. Let's find them analytically.

I'll focus on the solution with $0 \le f \le \sqrt{v}$, since the other solution can be found by sending $f \to -f$.

$$f' = \pm f\sqrt{v - f^2} \quad \Longrightarrow \quad \int \frac{df}{f\sqrt{v-f^2}} = \pm \xi$$

Using the indefinite integral $\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{arcsech}(x)$, we obtain

$$\pm \xi = -\frac{1}{\sqrt{v}} \operatorname{arcsech}\left(\frac{f}{\sqrt{v}}\right) \pm x_0$$ integration constant

$$\underset{\text{Invert}}{\Longrightarrow} \quad f = \sqrt{v}\, \operatorname{sech}\left[\mp\sqrt{v}(\xi - x_0)\right] = \sqrt{v}\, \operatorname{sech}\left[\sqrt{v}(\xi - x_0)\right]$$

Therefore the two travelling wave solutions are

$$u(x,t) = \pm\sqrt{v}\, \operatorname{sech}\left[\sqrt{v}(x - x_0 - vt)\right] .$$

$\boxed{\text{Ex } 13.3}$ "ϕ^4":

$$u_{tt} - u_{xx} + 2u(u^2-1) = 0$$

$$u_t,\ u_x,\ u+1 \xrightarrow[x\to-\infty]{} 0$$

$$u_t,\ u_x,\ u-1 \xrightarrow[x\to+\infty]{} 0$$

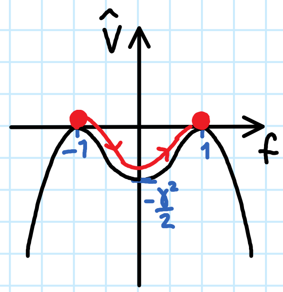
Sub in travelling wave $u(x,t) = f(x-vt) \equiv f(\xi)$:

$$(1-v^2)f'' = 2f(f^2-1) \iff f'' = 2\gamma^2 f(f^2-1) \quad , \quad \gamma \equiv \frac{1}{\sqrt{1-v^2}} .$$

Multiply by f' and integrate: $\frac{1}{2}(f')^2 = \gamma^2\left(\frac{f^4}{2} - f^2\right) + A$

$$\text{B.C.} \Rightarrow \quad 0 = \gamma^2\left(\tfrac{1}{2}-1\right) + A \quad \Rightarrow A = \frac{\gamma^2}{2} .$$

$$\text{So} \quad \frac{1}{2}(f')^2 \underbrace{- \frac{1}{2}\gamma^2(f^2-1)^2}_{=\hat{V}(f)} = 0$$



$$\Longrightarrow \quad f' = \pm\gamma(f^2-1) .$$

According to the b.c.'s, we look for a solution where f grows from -1 to $+1$. This selects

$$f' = \gamma(1-f^2) \quad \Longrightarrow \quad \gamma\xi = \int \frac{df}{1-f^2} = \operatorname{arctanh}(f) + \gamma x_0$$ integration constant

Therefore the travelling wave solution is

$$u(x,t) = \tanh\left[\gamma(x - x_0 - vt)\right] .$$

Ex 13.5

Burgers: $u_t + uu_x - u_{xx} = 0$

$u \to u_0, \quad u_x \to 0 \quad$ as $x \to -\infty$

$u \to u_1, \quad u_x \to 0 \quad$ as $x \to +\infty$ $\qquad (u_0 > u_1 > 0)$

Sub in travelling wave $u(x,t) = f(x - vt) \equiv f(\xi)$:

$$-vf' + ff' - f'' = 0$$

$$\int d\xi: \quad -vf + \tfrac{1}{2}f^2 - f' = A \qquad (*)$$

Impose B $\quad f \xrightarrow[\xi \to -\infty]{} u_0, \quad f \xrightarrow[\xi \to +\infty]{} u_1, \quad f' \xrightarrow[\xi \to \pm\infty]{} 0$:

$$A = \tfrac{1}{2}u_0(u_0 - 2v) = \tfrac{1}{2}u_1(u_1 - 2v)$$

$$\Rightarrow \overset{u_0+u_1}{\cancel{u_0^2 - u_1^2}} = 2v(\cancel{u_0 - u_1}) \Rightarrow v = \tfrac{1}{2}(u_0 + u_1)$$

$$\Rightarrow A = -\tfrac{1}{2}u_0 u_1$$

Integrate $(*)$ by separation of variables:

$$\int \frac{df}{f^2 - (u_0 + u_1)f + u_0 u_1} = 2\int d\xi$$

$$\int \frac{1}{(f - u_0)(f - u_1)} = 2(\xi - x_0)$$

$$\frac{1}{u_0 - u_1}\int df\left(\frac{1}{f - u_0} - \frac{1}{f - u_1}\right) = 2(\xi - x_0)$$

$$\frac{1}{u_0 - u_1}\ln\left|\frac{f - u_0}{f - u_1}\right| = 2(\xi - x_0)$$

Need $u_1 < f < u_0$, so

$$\frac{u_0 - f}{f - u_1} = \exp\left[\frac{2}{u_0 - u_1}(\xi - x_0)\right]$$

$$f(\xi) = \frac{u_0 + u_1 \exp\left[\frac{2}{u_0 - u_1}(\xi - x_0)\right]}{1 + \exp\left[\frac{2}{u_0 - u_1}(\xi - x_0)\right]}$$

$$\Rightarrow u(x,t) = \frac{u_0 + u_1 \exp\left[\frac{2}{u_0 - u_1}\left(x - x_0 - \tfrac{1}{2}(u_0 + u_1)t\right)\right]}{1 + \exp\left[\frac{2}{u_0 - u_1}\left(x - x_0 - \tfrac{1}{2}(u_0 + u_1)t\right)\right]}$$

CHECK:

$\xrightarrow[x \to -\infty]{} u_0$

$\xrightarrow[x \to +\infty]{} u_1$

Ex 15

1. $$F_n(\{\theta_m\}) = \underbrace{-a\sin\theta_n}_{\equiv F_n^{grav} = -\frac{\partial}{\partial\theta_n}V^{grav}} + \underbrace{\frac{1}{a}(\theta_{n+1}-\theta_n) + \frac{1}{a}(\theta_{n-1}-\theta_n)}_{\equiv F_n^{twist} = -\frac{\partial}{\partial\theta_n}V^{twist}} \equiv -\frac{\partial}{\partial\theta_n}V(\{\theta_m\})$$

This can be integrated to

$$V^{grav}(\{\theta_m\}) = a\sum_{m\in\mathbb{Z}}(1-\cos\theta_m)$$

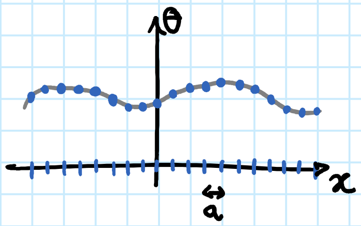
$$V^{twist}(\{\theta_m\}) = \frac{1}{2a}\sum_{m\in\mathbb{Z}}(\theta_{m+1}-\theta_m)^2$$

CHECK that partial derivatives reproduce the above forces!

where I've chosen the integration constants so that $V^{grav}=0$ when all pendula point down and $V^{twist}=0$ when all pendula are aligned. So

$$V(\{\theta_m\}) = a\sum_{n\in\mathbb{Z}}(1-\cos\theta_n) + \frac{1}{2a}\sum_{n\in\mathbb{Z}}(\theta_{n+1}-\theta_n)^2 .$$

2. In the CONTINUUM LIMIT, the set $\{\theta_n(t)\}$ approximates a continuous functions $\theta(x,t)$, s.t. $\theta(x=na,t)=\theta_n(t)$:



By the definition of the derivative as a limit

$$\frac{\theta_{n+1}(t)-\theta_n(t)}{a} \xrightarrow[a\to 0]{} \theta_x(x=na,t),$$

and by the definition of the integral as a limit of a sum

$$a\sum_{n\in\mathbb{Z}} f(\theta_n) \longrightarrow \int_{-\infty}^{+\infty} dx\, f(\theta(x,t))$$

In our case

$$V = a\sum_{n\in\mathbb{Z}}\left[(1-\cos\theta_n) + \frac{1}{2}\left(\frac{\theta_{n+1}-\theta_n}{a}\right)^2\right] \xrightarrow[a\to 0]{} \int_{-\infty}^{+\infty} dx\left[(1-\cos\theta(x,t)) + \frac{1}{2}\theta_x(x,t)^2\right]$$

Similarly

$$T = \frac{a}{2}\sum_n \overset{\frac{d}{dt}}{\dot{\theta}_n(t)^2} \xrightarrow[a\to 0]{} \int_{-\infty}^{+\infty} dx\,\frac{1}{2}\theta_t(x,t)^2 .$$

Ex 16

1. $$E = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{\lambda}{2}(u^2-a^2)^2 \right] \qquad (\lambda, a > 0)$$

$E < +\infty \Rightarrow u_t, u_x, u^2 - a^2 \longrightarrow 0$ as $x \to \pm\infty$.
therefore $u \longrightarrow a$ or $-a$.

2. There are 4 options according to the asymptotic values of u as $x \to \pm\infty$:

B.C. $u(-\infty,t)$	B.C $u(+\infty,t)$	Travelling wave sol'n $u(x,t)$
$-a$	$-a$	$-a$
$+a$	$+a$	$+a$
$-a$	$+a$	$+a \cdot \tanh[a\sqrt{\lambda}\,\gamma(x-x_0-vt)]$
$+a$	$-a$	$-a \cdot \tanh[a\sqrt{\lambda}\,\gamma(x-x_0-vt)]$

where the nontrivial travelling wave solutions

$$u(x,t) = \pm a \tanh\left[a\sqrt{\lambda}\,\gamma(x-x_0-vt)\right]$$

are obtained as in exercise 13.3 (the only difference being $a, \lambda \neq 1$).

The constant solutions have 0 energy, whereas for the nontrivial solutions

$$u_t^2 = a^4 \lambda \gamma^2 v^2 \operatorname{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)]$$
$$u_x^2 = a^4 \lambda \gamma^2 \cdot \operatorname{sech}^4[\quad " \quad]$$
$$\lambda(u^2-a^2)^2 = \lambda a^4 \cdot \operatorname{sech}^4[\quad " \quad]$$

using $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
and $\tanh^2 x - 1 = -\operatorname{sech}^2 x$

$$\Rightarrow E = \frac{1}{2}\int_{-\infty}^{+\infty} dx \left[u_t^2 + u_x^2 + \lambda(u^2-a^2)^2\right] = \frac{\lambda a^4}{2}\left(1+\gamma^2(1+v^2)\right)\int_{-\infty}^{\infty} dx \operatorname{sech}^4[a\sqrt{\lambda}\gamma(x-x_0-vt)]$$

$$= \frac{\sqrt{\lambda}a^3}{2\gamma} \cdot \underbrace{\frac{1-\cancel{v^2}+1+\cancel{v^2}}{1-v^2}}_{=2\gamma^2} \cdot \underbrace{\int_{-\infty}^{\infty} dy \operatorname{sech}^4(y)}_{=\frac{8}{6}=\frac{4}{3}} = \frac{4}{3} a^3 \sqrt{\lambda} \cdot \gamma \quad , \qquad \text{with } \gamma = \frac{1}{\sqrt{1-v^2}}.$$

Static solutions $(v=0 \Leftrightarrow \gamma=1)$ have the lowest energy: $M = E\big|_{v=0} = \frac{4}{3}a^3\sqrt{\lambda}$.

3.
$$E = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] = \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[u_t^2 + \left(u_x \mp \sqrt{\lambda} (u^2 - a^2) \right)^2 \pm 2\sqrt{\lambda}\, u_x (u^2 - a^2) \right]$$
$$= \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[u_t^2 + \left(u_x \mp \sqrt{\lambda} (u^2 - a^2) \right)^2 \right] \pm \sqrt{\lambda} \left[\frac{u^3}{3} - a^2 u \right]_{-\infty}^{+\infty}$$
$$\geq \pm \sqrt{\lambda} \left[\frac{u^3}{3} - a^2 u \right]_{-\infty}^{+\infty}$$

For the "kink" $\lim_{x \to \pm\infty} u = \pm a$, therefore

$$E \geq \pm \sqrt{\lambda} \cdot 2 \cdot \left(\frac{a^3}{3} - a^2 \cdot a \right) = \mp \frac{4}{3} \sqrt{\lambda}\, a^3 \implies E \geq \frac{4}{3} \sqrt{\lambda}\, a^3 .$$

$$E = \frac{4}{3} \sqrt{\lambda}\, a^3 \quad \text{iff} \quad u_t = 0 \quad \text{and} \quad u_x + \sqrt{\lambda} (u^2 - a^2) = 0 .$$

Integrating, $\displaystyle \int \frac{du}{a^2 - u^2} = \sqrt{\lambda} \int dx \implies \frac{1}{a} \operatorname{arctanh} \frac{u}{a} = \sqrt{\lambda} (x - x_0)$

$$\implies u(x, \cancel{t}) = a \tanh \left[a\sqrt{\lambda} (x - x_0) \right],$$

which is indeed the static kink found in part 2.

Ex 18

$x \in [-\pi/2, \pi/2]$ $\qquad E = \int_{-\pi/2}^{\pi/2} dx \, \frac{1}{2}\left(u_t^2 + u_x^2 + 1 - u^2\right)$

$u(x,t)$ satisfies $|u(x,t)| \le 1$ and BC $|u(\pm\frac{\pi}{2}, t)| = 1$.

$$E \geq \frac{1}{2}\int_{-\pi/2}^{\pi/2} dx \left(u_x \pm \sqrt{1-u^2}\right)^2 \mp \int_{-\pi/2}^{\pi/2} dx \, u_x \sqrt{1-u^2} \qquad (= \text{ iff } u_t = 0)$$

$$\geq \mp \int_{u(-\frac{\pi}{2},t)}^{u(\frac{\pi}{2},t)} du \sqrt{1-u^2} \qquad (= \text{ iff } u_x = \mp\sqrt{1-u^2})$$

$$\geq \mp \frac{1}{2}\left[u\sqrt{1-u^2} + \arcsin u\right]_{u(-\frac{\pi}{2},t)}^{u(\frac{\pi}{2},t)}$$

"Kink": $u(-\frac{\pi}{2},t) = -1$, $u(+\frac{\pi}{2},t) = +1$

$$[\dots]_{-1}^{1} = \arcsin(1) - \arcsin(-1) = \pi .$$

Hence $E \geq \frac{\pi}{2}$, picking the lower sign.

The bound is saturated if $u_t = 0$ and $u_x = \sqrt{1-u^2}$. Integrating,

$$\int \frac{du}{\sqrt{1-u^2}} = \int dx \quad \Rightarrow \quad \arcsin u = x - x_0$$

Impose BC: $\begin{cases} \arcsin(-1) = -\frac{\pi}{2} - x_0 \\ \arcsin(1) = +\frac{\pi}{2} - x_0 \end{cases} \Rightarrow x_0 = 0$

$\Rightarrow \quad u(x) = \sin x$ saturates the lower bound.

CHECK: $E = \frac{1}{2}\int_{-\pi/2}^{\pi/2} dx \left[u_t^2 + u_x^2 + 1 - u^2\right] = \frac{1}{2}\int_{-\pi/2}^{\pi/2} dx \left[0 + \cos^2 x + 1 - \sin^2 x\right] = \int_{\pi/2}^{-\pi/2} dx \, \frac{1+\cos 2x}{2} = \frac{\pi}{2}.$

Ex 20.1

Sine-Gordon kink of velocity v: $u(x,t) = 4\arctan\left(e^{\gamma(x-x_0-vt)}\right)$.

Up to normalization, the TOPOLOGICAL CHARGE is

$$Q_0 \propto \frac{1}{2\pi}\left[u\right]_{-\infty}^{+\infty} = \frac{2\pi - 0}{2\pi} = 1 .$$

(Any normalization is fine. I just wanted to see $Q_0 \propto [u]_{-\infty}^{+\infty}$.)

The ENERGY is

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + (1-\cos u)\right] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(1+v^2)\,u_x^2 + 2\sin^2\frac{u}{2}\right]$$

$$= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}(1+v^2)\cdot\gamma^2\; 4\,\mathrm{sech}^2\left(\gamma(x-x_0-vt)\right) + 2\sin^2\left(2\arctan\left(e^{\gamma(x-x_0-vt)}\right)\right)\right]$$

$$= \frac{1}{\gamma}\int_{-\infty}^{\infty} dy \left[2\,\frac{1+v^2}{1-v^2}\,\mathrm{sech}^2(y) + 8\sin^2\left(\arctan(e^y)\right)\cos^2\left(\arctan(e^y)\right)\right]$$

Let $\theta \equiv \arctan(e^y)$. Then the last term is:

$$8\sin^2\theta\cdot\cos^2\theta = 8\tan^2\theta\cdot\cos^4\theta = 8\,\frac{\tan^2\theta}{(\tan^2\theta+1)^2} = 2\left(\frac{2}{\tan\theta + \frac{1}{\tan\theta}}\right)^2 = 2\cdot\left(\frac{2}{e^y + e^{-y}}\right)^2 = 2\,\mathrm{sech}^2(y) .$$

$$\Rightarrow\quad E = \frac{1}{\gamma}\int_{-\infty}^{+\infty} dy\; 2\left(\frac{1+v^2}{1-v^2}+1\right)\mathrm{sech}^2(y) = 4\gamma\int_{-\infty}^{\infty} dy\;\mathrm{sech}^2(y) = 8\gamma = \frac{8}{\sqrt{1-v^2}} .$$

The MOMENTUM is

$$P = -\int_{-\infty}^{+\infty} dx\; u_t u_x = v\int_{-\infty}^{\infty} dx\; u_x^2 = 4v\gamma^2\int_{-\infty}^{\infty} dx\;\mathrm{sech}^2\left(\gamma(x-x_0-vt)\right) = 4\gamma v\int_{-\infty}^{\infty} dy\;\mathrm{sech}^2(y) = 8\gamma v.$$

All these 3 charges are conserved (t-independent).

In terms of the mass $M = E\big|_{v=0} = 8$,

$$E = M\gamma = \frac{M}{\sqrt{1-v^2}} = M + \frac{1}{2}Mv^2 + O(Mv^4)$$

$$P = M\gamma v = Mv + O(Mv^3) .$$

Ex 23

1. For ρ_1 and ρ_2, see the lecture notes. For ρ_*:

$$(xu-3tu^2)_t = xu_t - 3u^2 - 6tuu_t \underset{\text{KdV}}{=} -x(6uu_x+u_{xxx}) - 3u^2 + 6tu(6uu_x+u_{xxx})$$

Term by term:

$$-6xuu_x = -(3xu^2)_x + 3u^2$$
$$-xu_{xxx} = -(xu_{xx})_x + u_{xx} = (-xu_{xx}+u_x)_x$$
$$-3u^2$$
$$36tu^2u_x = (12tu^3)_x$$
$$6tuu_{xxx} = (6tuu_{xx})_x - 6tu_xu_{xx} = (6tuu_{xx}-3tu_x^2)_x$$

$$\Rightarrow (\rho_*)_t = (xu-3tu^2)_t = \left(-3xu^2 - xu_{xx} + u_x + 12tu^3 + 6tuu_{xx} - 3tu_x^2\right)_x \equiv -(j_*)_x ,$$

and assuming that u and its x-derivatives tend to 0 fast enough as $x\to\pm\infty$,

$$Q_* = \int_{-\infty}^{\infty} dx\,(xu-3tu^2)$$

is a conserved charge like Q_1 and Q_2.

(More precisely: the usual b.c.'s imply that $u, u_x, u_{xx}, \ldots \to 0$ as $|x|\to\infty$ $\forall t$.
In order to have a conserved charge Q_*, however, we need $\lim\limits_{x\to+\infty} j_* = \lim\limits_{x\to-\infty} j_*$ and Q_* finite.
For the latter, we need $u\to 0$ faster than x^{-2}, unless there are cancellations.)

2. $$u_{\mu,x_0}(x,t) = 2\mu^2\,\mathrm{sech}^2\left[\mu(x-x_0-4\mu^2t)\right]$$

(I will use $\displaystyle\int_{-\infty}^{+\infty} dx\,\mathrm{sech}^{2n}x = \frac{2^{2n-1}(n-1)!^2}{(2n-1)!}$.)

$$Q_1 = \int_{-\infty}^{+\infty} dx\, u_{\mu,x_0}(x,t) = 2\mu^2\int_{-\infty}^{+\infty} dx\,\mathrm{sech}^2(\mu x) \underset{y=\mu x}{=} 2\mu\int_{-\infty}^{+\infty} dy\,\mathrm{sech}^2 y = 2\mu\cdot 2 = 4\mu$$

$$Q_2 = \int_{-\infty}^{+\infty} dx\, u^2_{\mu,x_0}(x,t) = 4\mu^4\int_{-\infty}^{+\infty} dx\,\mathrm{sech}^4(\mu x) \underset{y=\mu x}{=} 4\mu^3\int_{-\infty}^{+\infty} dy\,\mathrm{sech}^4 y = 4\mu^3\cdot\frac{2^3}{3!} = \frac{16}{3}\mu^3 .$$

$$Q_* = \int_{-\infty}^{+\infty} dx \left[x\, u_{\mu,x_0} - 3t\, u^2_{\mu,x_0} \right] = 2\mu^2 \int_{-\infty}^{+\infty} dx\, x \cdot \mathrm{sech}^2\left[\mu(x - x_0 - 4\mu^2 t)\right] - 3t\, Q_2$$

NOTE: integrable because $\mathrm{sech}^2 x \xrightarrow[|x|\to\infty]{} 0$ exponentially

$$= -3t\, Q_2 + 2\mu^2 \left\{ \int_{-\infty}^{+\infty} dx\, (x - x_0 - 4\mu^2 t)\, \mathrm{sech}^2\left[\mu(x - x_0 - 4\mu^2 t)\right] + (x_0 + 4\mu^2 t) \int_{-\infty}^{+\infty} dx\, \mathrm{sech}^2\left[\mu(x - x_0 - 4\mu^2 t)\right] \right\}$$

0 since it's the integral of an odd (and integrable) function

$$= (x_0 + 4\mu^2 t)\, Q_1 - 3t\, Q_2$$

$$= (x_0 + \cancel{4\mu^2 t}) \cdot 4\mu - \cancel{3t \cdot \frac{16}{3}\mu^2} = 4\mu x_0,$$

which indeed does not depend on time.

SUMMARY: for the 1-soliton $u_{\mu,x_0}(x,t) = 2\mu^2\, \mathrm{sech}^2\left[\mu(x - x_0 - 4\mu^2 t)\right]$,

$Q_1 = 4\mu$, $Q_2 = \frac{16}{3}\mu^3$ and $Q_* = 4\mu x_0$ are conserved charges.

3. Since Q_1 & Q_2 are conserved, they must have the same values for the initial condition $u(x, t=0)$ and for the late time solution $u(x, t\to+\infty)$ which consists of two well-separated solitons.

$t=0$: $u(x,0) = 6\, \mathrm{sech}^2 x$

$$Q_1 = \int_{-\infty}^{+\infty} dx\, u(x,0) = 6 \int_{-\infty}^{+\infty} dx\, \mathrm{sech}^2 x = 12$$

$$Q_2 = \int_{-\infty}^{+\infty} dx\, u^2(x,0) = 36 \int_{-\infty}^{+\infty} dx\, \mathrm{sech}^4 x = 36 \cdot \frac{4}{3} = 48 .$$

$t\to+\infty$: $u(x,t) \approx u_{\mu_1,x_1}(x,t) + u_{\mu_2,x_2}(x,t)$ (i.e. $\lim_{t\to+\infty} (\mathrm{LHS} - \mathrm{RHS}) = 0$)

Sum of two well-separated solitons. Charges add up:

$$Q_1 = 4\mu_1 + 4\mu_2 = 4(\mu_1 + \mu_2)$$

$$Q_2 = \frac{16}{3}(\mu_1^3 + \mu_2^3) .$$

Equating the (conserved) charges of the solution at $t=0$ and $t\to+\infty$ we find:

$$\begin{cases} 4(\mu_1+\mu_2) = 12 \\ \frac{16}{3}(\mu_1^3+\mu_2^3) = 48 \end{cases} \Rightarrow \begin{cases} \mu_1+\mu_2 = 3 \\ \cancel{(\mu_1+\mu_2)}(\mu_1^2-\mu_1\mu_2+\mu_2^2) = \cancel{9}\,3 \end{cases} \Rightarrow \begin{cases} \mu_1+\mu_2 = 3 \\ (\mu_1+\mu_2)^2 - 3\mu_1\mu_2 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} \mu_1+\mu_2 = 3 \\ \mu_1\mu_2 = 2 \end{cases} \Rightarrow \quad (\mu_1,\mu_2) = (1,2) \text{ or } (2,1) .$$

Therefore the velocities of the two solitons which appear at late times are $v_i = 4\mu_i^2 = 4, 16$, and their heights are $2\mu_i^2 = 2, 8$, in agreement with the animations seen in the 1st week.
file N=2

4. Same logic as above, now for

$$u(x,t) \underset{t\to-\infty}{\approx} u_{\mu_1,x_1}(x,t) + u_{\mu_2,x_2}(x,t)$$

$$u(x,t) \underset{t\to+\infty}{\approx} u_{\mu_1,y_1}(x,t) + u_{\mu_2,y_2}(x,t)$$

and the conserved charge Q_*:

$$\left.Q_*\right|_{t\to-\infty} = 4(\mu_1 x_1 + \mu_2 x_2)$$
$$\shortparallel$$
$$\left.Q_*\right|_{t\to+\infty} = 4(\mu_1 y_1 + \mu_2 y_2)$$

$$\Rightarrow \quad \mu_1(y_1-x_1) = -\mu_2(y_2-x_2) \quad \Rightarrow \quad y_2 - x_2 = -\frac{\mu_1}{\mu_2}(y_1-x_1) .$$

Ex 36

1. $D_t^m D_x^n$ is bilinear because

$$(\partial_t-\partial_{t'})^m(\partial_x-\partial_{x'})^n = \sum_{h=0}^{m}\binom{m}{h}(-1)^{m-h}\partial_t^h\partial_{t'}^{m-h}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\partial_x^k\partial_{x'}^{n-k}$$

$$= \sum_{h=0}^{m}\sum_{k=0}^{n}\binom{m}{h}\binom{n}{k}(-1)^{m+n-h-k}\left(\partial_t^h\partial_x^k\right)\left(\partial_{t'}^{m-h}\partial_{x'}^{n-k}\right)$$

is a linear combination of linear differential operators in (t,x) $\partial_t^h\partial_x^k$ multiplied by linear differential operators in (t',x') $\partial_{t'}^{m-h}\partial_{x'}^{n-k}$. Indeed

$$\partial_t^h\partial_x^k\left(a_1f_1(x,t)+a_2f_2(x,t)\right) = a_1\,\partial_t^h\partial_x^k f_1(x,t)+a_2\,\partial_t^h\partial_x^k f_2(x,t)$$

$$\partial_{t'}^{m-h}\partial_{x'}^{n-k}\left(a_1g_1(x',t')+a_2g_2(x',t')\right) = a_1\,\partial_{t'}^{m-h}\partial_{x'}^{n-k}g_1(x',t')+a_2\,\partial_{t'}^{m-h}\partial_{x'}^{n-k}g_2(x',t')$$

implies that

$$D_t^m D_x^n(a_1f_1+a_2f_2,g) = a_1 D_t^m D_x^n(f_1,g)+a_2 D_t^m D_x^n(f_2,g)$$

$$D_t^m D_x^n(f,a_1g_1+a_2g_2) = a_1 D_t^m D_x^n(f,g_1)+a_2 D_t^m D_x^n(f,g_2).$$

2. $$D_t^m D_x^n(f,g) = (\partial_t-\partial_{t'})^m(\partial_x-\partial_{x'})^n f(x,t)\,g(x',t')\Big|_{\substack{x'=x\\t'=t}}$$

$$= (-1)^{m+n}(\partial_{t'}-\partial_t)^m(\partial_{x'}-\partial_x)^n g(x',t')\,f(x,t)\Big|_{\substack{x'=x\\t'=t}} = (-1)^{m+n}D_t^m D_x^n(g,f).$$

3. $$D_t^2(f,g) = (\partial_t-\partial_{t'})^2 f(x,t)\,g(x',t')\Big|_{\substack{x'=x\\t'=t}} = \left[f_{tt}(x,t)g(x',t')-2f_t(x,t)g_{t'}(x',t')+f(x,t)g_{t't'}(x',t')\right]\Big|_{\substack{x'=x\\t'=t}}$$

$$= f_{tt}\,g-2f_tg_t+fg_{tt}.$$

$$D_x^4(f,g) = (\partial_x-\partial_{x'})^4 f(x,t)\,g(x',t')\Big|_{\substack{x'=x\\t'=t}} = f_{xxxx}\,g-4f_{xxx}\,g_x+6f_{xx}\,g_{xx}-4f_x\,g_{xxx}+fg_{xxxx}.$$

Ex 38

1. $\theta_i = a_i x + b_i t + c_i$. ($a_i, b_i, c_i$ constants)

$$D_t D_x (e^{\theta_1}, e^{\theta_2}) = (\partial_t - \partial_{t'})(\partial_x - \partial_{x'}) e^{\theta_1 + \theta_2'} \Big|_{\substack{x'=x \\ t'=t}} ,$$

where $\theta_2' = a_2 x' + b_2 t' + c_2$. Differentiating the exponentials

$$(\partial_t - \partial_{t'})(\partial_x - \partial_{x'}) e^{\theta_1 + \theta_2'} = (b_1 - b_2)(a_1 - a_2) e^{\theta_1 + \theta_2'}$$

and taking $(x', t') \to (x, t)$

$$D_t D_x (e^{\theta_1}, e^{\theta_2}) = (b_1 - b_2)(a_1 - a_2) e^{\theta_1 + \theta_2} .$$

2. Similarly,

$$(\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n e^{\theta_1 + \theta_2'} = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^{n-1} (a_1 - a_2) e^{\theta_1 + \theta_2'}$$

$$= (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^{n-2} (a_1 - a_2)^2 e^{\theta_1 + \theta_2'} = \ldots = (a_1 - a_2)^n (\partial_t - \partial_{t'})^m e^{\theta_1 + \theta_2'}$$

$$\underset{(\text{similarly})}{=} \ldots = (a_1 - a_2)^n (b_1 - b_2)^m e^{\theta_1 + \theta_2'}$$

$$\Rightarrow D_t^m D_x^n (e^{\theta_1}, e^{\theta_2}) = (b_1 - b_2)^m (a_1 - a_2)^n e^{\theta_1 + \theta_2} .$$

Ex 42

Boussinesq eqn: $u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0$.

Sub in $u = 2 \frac{\partial^2}{\partial x^2} \log f$ and integrate twice in x:

integration constants

$$2(\log f)_{tt} - 2(\log f)_{xx} - 3 \cdot 4 (\log f)_{xx}^2 - 2(\log f)_{xxxx} = a(t) x + b(t)$$

We are free to redefine

$$\log f \longrightarrow \log f + \alpha(t) x + \beta(t)$$

without changing u. Choosing $\alpha(t), \beta(t)$ s.t.

$$2\alpha''(t) = a(t)$$
$$2\beta''(t) = b(t)$$

we can absorb the RHS. Once that is done, the equation becomes

$$2(\log f)_{tt} - 2(\log f)_{xx} - 3\cdot 4(\log f)_{xx}^2 - 2(\log f)_{xxxx} = 0 \qquad (*)$$

The terms are:

$$2(\log f)_{tt} = 2\partial_t \frac{f_t}{f} = 2\left(\frac{f_{tt}}{f} - \frac{f_t^2}{f^2}\right) = 2\,\frac{f_{tt}f - f_t^2}{f^2} = \frac{D_t^2(f,f)}{f^2}$$

$$2(\log f)_{xx} = 2\,\frac{f_{xx}f - f_x^2}{f^2} = \frac{D_x^2(f,f)}{f^2}$$

$$\Rightarrow 3\cdot 4(\log f)_{xx}^2 = 12\,\frac{f_{xx}^2 f^2 - 2f_{xx}ff_x^2 + f_x^4}{f^4}$$

$$2(\log f)_{xxx} = 2\,\frac{f_{xxx}f + f_{xx}f_x - 2f_xf_{xx}}{f^2} - 4\,\frac{(f_{xx}f - f_x^2)f_x}{f^3} = 2\,\frac{f_{xxx}f^2 - 3ff_xf_{xx} + 2f_x^3}{f^3}$$

$$\Rightarrow 2(\log f)_{xxxx} = 2\,\frac{f_{xxxx}f^2 + 2f_{xxx}ff_x - 3f_x^2f_{xx} - 3ff_{xx}^2 - 3ff_xf_{xxx} + 6f_x^2f_{xx}}{f^3} - 6\,\frac{(f_{xxx}f^2 - 3ff_xf_{xx} + 2f_x^3)f_x}{f^4}$$

$$= \frac{2}{f^4}\Big[f_{xxxx}f^3 + 2f^2f_xf_{xxx} - 3ff_x^2f_{xx} - 3f^2f_{xx}^2 - 3f^2f_xf_{xxx} + 6ff_x^2f_{xx}$$
$$-3f_{xxx}f^2f_x + 9ff_x^2f_{xx} - 6f_x^4\Big]$$

$$= \frac{2}{f^4}\left[-6f_x^4 + 12ff_x^2f_{xx} - 3f^2f_{xx}^2 - 4f^2f_xf_{xxx} + f^3f_{xxxx}\right]$$

$$\Rightarrow 3\cdot 4(\log f)_{xx}^2 + 2(\log f)_{xxxx} = \frac{2}{f^4}\left(3f^2f_{xx}^2 - 4f^2f_xf_{xxx} + f^3f_{xxxx}\right) = \frac{2}{f^2}\left(f_{xxxx} - 4f_{xxx}f_x + 3f_{xx}^2\right)$$
$$= \frac{D_x^4(f,f)}{f^2}.$$

Therefore, multiplying the eqn (*) by f^2 we can obtain an equation in bilinear form

$$(D_t^2 - D_x^2 - D_x^4)(f,f) = 0.$$