SOLUTIONS TO SELECTED EXERCISES

Ex 1

Sub $u(x,t) = \dfrac{2}{\cosh^2(x-vt)}$ in KdV eqn $u_t + 6uu_x + u_{xxx} = 0$:

$$u_t = \frac{4v}{\cosh^3(x-vt)} \cdot \sinh(x-vt)$$

$$u_x = \frac{-4}{\cosh^3(x-vt)} \sinh(x-vt)$$

$$u_{xx} = \frac{12}{\cosh^4(x-vt)} \sinh^2(x-vt) - \frac{4}{\cosh^2(x-vt)} \underset{\cosh^2 x - \sinh^2 x = 1}{=} \frac{8}{\cosh^2(x-vt)} - \frac{12}{\cosh^4(x-vt)}$$

$$u_{xxx} = \left[\frac{-16}{\cosh^3(x-vt)} + \frac{48}{\cosh^5(x-vt)} \right] \sinh(x-vt)$$

So KdV $\Rightarrow$ $0 = \dfrac{4\sinh(x-vt)}{\cosh^3(x-vt)} \cdot \left[v - \cancel{\dfrac{48}{\cosh^2(x-vt)}} - 4 + \cancel{\dfrac{48}{\cosh^2(x-vt)}} \right]$.

Therefore $u = \dfrac{2}{\cosh^2(x-vt)}$ is a solution of KdV if (and only if) $\underline{\underline{v = 4}}$.

Ex 2

1. Let $u(x,t) = A\,g(X,T)$, where $X = Bx$ and $T = Ct$.

Chain rule $\Rightarrow$ $u_t(x,t) = AC\,g_T(X,T)$, $u_x(x,t) = AB\,g_X(X,T)$, $u_{xxx}(x,t) = AB^3 g_{XXX}(X,T)$

$$\Rightarrow \quad u_t + 6uu_x + u_{xxx} = AC\,g_T(X,T) + 6A^2B\,g(X,T)\,g_X(X,T) + AB^3 g_{XXX}(X,T)$$

$$= AC \cdot \left[g_T(X,T) + \frac{AB}{C} \cdot 6g(X,T)g_X(X,T) + \frac{B^3}{C} g_{XXX}(X,T) \right].$$

If $g(X,T)$ solves KdV (in its variables X and T !), then $u(x,t)$ solves KdV (in its variables x and t) provided that $\dfrac{AB}{C} = \dfrac{B^3}{C} = 1$, because in that case

$$u_t + 6uu_x + u_{xxx} \underset{\frac{AB}{C} = \frac{B^3}{C} = 1}{=} AC\,\underbrace{\left[g_T(X,T) + 6g(X,T)g_X(X,T) + g_{XXX}(X,T) \right] = 0}_{\text{KdV for } g(X,T)}.$$

$$\frac{AB}{C} = \frac{B^3}{C} = 1 \quad \Leftrightarrow \quad \underline{A = B^2, \; C = B^3}.$$

2. $u(x,t) = g(x,t) \equiv \dfrac{2}{\cosh^2(x-4t)}$ 1-soliton solution of KdV

$$\Rightarrow u(x,t) = B^2 g(Bx, B^3 t) = \frac{2B^2}{\cosh^2[B(x-4B^2t)]} \equiv u_B(x,t)$$

label, not derivative!

1-parameter family of 1-soliton sol'ns (with parameter B)

3. $(\text{Height of } u_B) = 2B^2$
$(\text{Velocity of } u_B) = 4B^2$ $\Longrightarrow$ Velocity $= 2$ Height.

The "width" is a measure of how the lump is concentrated in space (x). Since the dependence of u_B on x is only through Bx, we can say that $(\text{Width of } u_B) \propto \frac{1}{B}$. The precise proportionality factor depends on the precise definition of width that you might choose, but regardless of that choice

(B>0 with no loss of generality)

$$v \longrightarrow 4v = B^2 v \Longrightarrow \text{width} \longrightarrow \frac{1}{B}\,\text{width} = \frac{\text{width}}{2}.$$

Don't worry if you were confused by part 3 because I didn't define the width unambiguously. My aim was to make you think about the meaning of width (and see how many of you would realize that the answer was largely independent of the def'n).

I might run a few more social experiments like this during the year, but not in the exam, I promise. And feel free to ask me questions about the homework if you think you need more information than is given in the text.

SUMMARY of this exercise: For 1-soliton solutions of KdV

$$\text{Height} \propto \text{Velocity} \propto \text{Width}^{-2}.$$

Ex 3

Direct method: check using the chain rule.

- $$v_t + 6\varepsilon v v_x + v_{xxx} = \frac{1}{\varepsilon}u_t + 6\varepsilon\frac{1}{\varepsilon^2}u u_x + \frac{1}{\varepsilon}u_{xxx} = \frac{1}{\varepsilon}(u_t + 6uu_x + u_{xxx}) = 0.$$

- $$w_t + 6ww_x + \varepsilon w_{xxx} = \varepsilon \underbrace{\frac{\partial T}{\partial t}}_{=\varepsilon}\frac{\partial u(x,T)}{\partial T}\bigg|_{T=\varepsilon t} + 6\varepsilon^2\, u(x,T)u_x(x,T)\bigg|_{T=\varepsilon t}$$
$$+ \varepsilon\cdot\varepsilon\, u_{xxx}(x,T)\bigg|_{T=\varepsilon t} = \varepsilon^2\left[u_T(x,T) + 6u(x,T)u_x(x,T) + u_{xxx}(x,T)\right]\bigg|_{T=\varepsilon t} = 0.$$

Alternatively, suppose you didn't know the final eqn for v, w and wanted to derive it. Express u in terms of v (/w) and sub in KdV eqn:

- $u(x,t) = \varepsilon v(x,t)$: KdV $\Rightarrow$ $\varepsilon v_t + 6\varepsilon^2 v v_x + \varepsilon v_{xxx} = 0$
  $\Rightarrow$ $v_t + 6\varepsilon v v_x + v_{xxx} = 0$.

- $u(x,\varepsilon t) = \frac{1}{\varepsilon}w(x,t)$: KdV for $u(x,\varepsilon t)$ is

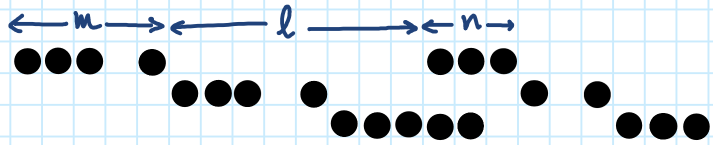
$$0 = \frac{\partial u(x,\varepsilon t)}{\partial(\varepsilon t)} + 6u(x,\varepsilon t)u_x(x,\varepsilon t) + u_{xxx}(x,\varepsilon t)$$

from $u(x,\varepsilon t) = \frac{1}{\varepsilon}w(x,t)$

from $\frac{\partial t}{\partial(\varepsilon t)}$

$$= \frac{1}{\varepsilon}\cdot\frac{1}{\varepsilon}w_t(x,t) + \frac{6}{\varepsilon^2}w(x,t)w_x(x,t) + \frac{1}{\varepsilon}w_{xxx}(x,t)$$
$$= \frac{1}{\varepsilon^2}\left[w_t + 6ww_x + \varepsilon w_{xxx}\right]$$

Ex 4



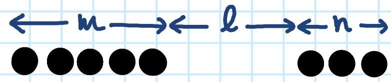
When the length m soliton is far enough to the left of the length n soliton (with $m>n$), we can evolve the system forward by one unit of time, to reduce the separation ℓ:

$$\ell \longrightarrow \ell - m + n .$$

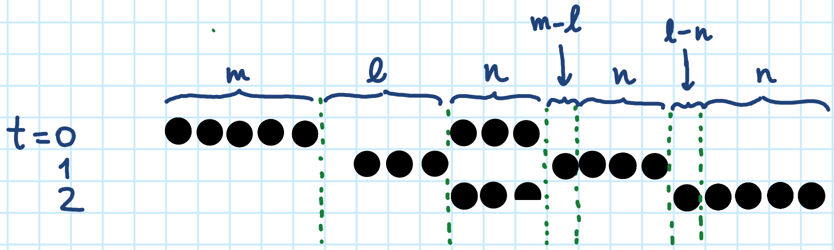
We can iterate this until the separation ℓ reaches the range

$$n \leq \ell < m,$$

and take this as our starting point:



Let's evolve this configuration forward and see what happens to the two solitons. To cover all cases, we'll have to evolve the system by $\Delta t = 2$.



Make sure you understand how this follows from the ball and box rule.
If you don't, ask me.

Unless $n=\ell$, at $t=1$ the system is in an intermediate configuration, still in the middle of the interaction. But already at $t=2$ the faster soliton has overtaken the slower, after which they keep travelling undisturbed to the right, with increasing separation.

To calculate the phase shifts of the two solitons, let's compare their positions at $t=2$ (after the interaction) with the positions they would have had at the same time had the other soliton not been there.

Compared to $t=0$, the length m soliton has moved to the right by

$$m+\ell+n+(m-\ell)+n = 2m+2n$$

boxes. It would have moved by $2m$ boxes in the absence of an interaction, so its phase shift is $+2n$.

Compared to $t=0$, the length n soliton has moved to the right by 0 boxes.
It would have moved by $2n$ boxes had no interaction taken place, therefore its phase shift is $-2n$.

CONCLUSION :

The two solitons emerge from the interaction with the same lengths and velocities, but advanced/delayed by

$$|\text{Phase shift}| = 2\times(\text{Length of the slower soliton}).$$

- What can go "wrong" if $\ell < n$?

The starting point is in the middle of the interaction! See for instance $t=1$ in the picture above. This is just an intermediate configuration, which can involve different objects than the solitons appearing in the far past/future.

Ex 5

To make my life easier and aid memory, I will avoid colours and use instead "1" for the (blue) ball which is moved first and "2" for the (red) ball which is moved afterwards.

E.g. 

1 1 2 ↕ 2 2 1   2   1 2

- Clearly:

  A row of n consecutive 1's is a speed n soliton

  " " " " " 2's " " " " "

because if a single colour/type of ball is present, it evolves as in the single colour ball and box model.

```
111                        2222
    111                        2222
        111                        2222
```

- Next, consider a sequence/row where 1's are separated by 2's, e.g.

```
1121221
   2   211112
    2   2     11112
```

Since 1's move first, after one unit of time all 1's will be consecutive, so this is not a soliton. In order to have a soliton, all the 1's must be consecutive!

- Consider then a row of m 2's followed by n-m 1's:

$$\underbrace{1111}_{n-m}\underbrace{222}_{m}$$

```
1111222
       1111222
```

1's move first, translating by n boxes to the right, and 2's follow.

⟶ Speed n soliton.

This works for $m = 0, 1, 2, \dots, n$ $\Rightarrow$ <u>n+1 speed n</u> solitons.

- What if we have m 1's followed by $n-m$ 2's ?

$$\overbrace{2222}^{n-m\,>}\overbrace{111}^{m}$$
$$\quad\;\; 2221112$$

$$\overbrace{222}^{n-m\,=}\overbrace{111}^{m}$$
$$\quad\;\; 222111$$

$$\overbrace{22}^{n-m\,<}\overbrace{111}^{m}$$
$$\quad\;\; 22\;\;111$$

1's move first by m units. 2's follow:

1) $n-m>m$: at least one 2 overtakes the row of 1's.

2) $n-m=m$: final configuration = initial configuration.
$\rightarrow$ Speed $m=\frac{n}{2}$ soliton (if $n\in 2\mathbb{Z}$).

3) $n-m<m$: the row of 2's is left behind.

- Finally, what about

$$\overbrace{2222}^{a}\overbrace{111}^{b}\overbrace{22222}^{c} \quad ?$$

1's move first, leaving a gap between the two rows of 2's. The gap is (partially or completely) filled by some 2's from the left group, but anyway the 2's from the right group leave a gap of (at least) c empty boxes to the left of the row of 1's.

$$222\underbrace{\qquad\qquad}_{c}111222222 \qquad \rightarrow \text{Not a soliton!}$$

SUMMARY:

- n is odd : there are $n+1$ length n solitons, all of speed n.

$$\overbrace{1111}^{n-m}\overbrace{222}^{m} \qquad\qquad m=0,1,2,\dots,n.$$

- n is even : in addition to the above, there's an extra length n soliton of speed $\frac{n}{2}=m$.

$$\overbrace{222}^{m}\overbrace{111}^{m}$$
$$\qquad\quad 222111$$

In fact, this latter soliton is better thought of as the union of two consecutive length $\frac{n}{2}$ and speed $\frac{n}{2}$ solitons.

Indeed, consider the scattering (→ Ex 7)

```
speed 3            speed 2
 122                2211
         122            2211
                122          2211
                     12          22112
                          12         22    112
                               12        22      112
                                   12        22          112
                                speed 2   speed 2      speed 3
```

Note that numbers (/colours) are reshuffled, but lengths and speeds are unchanged.
The length $2m$ and speed m soliton breaks up into two length m and speed m solitons.

→ Look at Ex 7 to explore more properties.

Ex 10

1. $$\int_{-\infty}^{+\infty} dk\, e^{-A(k-\bar{k})^2+iBk} = e^{iB\bar{k}}\int_{-\infty}^{+\infty} dk\, e^{-A(k-\bar{k})^2+iB(k-\bar{k})} \underset{k=\bar{k}+h}{=} e^{iB\bar{k}}\int_{-\infty}^{+\infty} dh\, e^{-Ah^2+iBh}$$

$$= e^{iB\bar{k}}\int_{-\infty}^{+\infty} dk\, e^{-A\left(h-i\frac{B}{2A}\right)^2}\cdot e^{-\frac{B^2}{4A}} = \sqrt{\frac{\pi}{A}}\, e^{iB\bar{k}}\, e^{-B^2/4A}$$

2. $$z(x,t) = \int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2+i[kx-\omega(k)t]}$$

$$\approx e^{i[\bar{k}x-\omega(\bar{k})t]}\int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2+i(k-\bar{k})x-i\omega'(\bar{k})(k-\bar{k})t-\frac{i}{2}\omega''(\bar{k})(k-\bar{k})^2t}$$

$$= e^{i[\bar{k}x-\omega(\bar{k})t]}\int_{-\infty}^{+\infty} dk\, e^{-\left[a^2+\frac{i}{2}\omega''(\bar{k})t\right](k-\bar{k})^2+i\left[x-\omega'(\bar{k})t\right](k-\bar{k})}$$

$$= e^{i[\bar{k}x-\omega(\bar{k})t]}\sqrt{\frac{\pi}{a^2+\frac{i}{2}\omega''(\bar{k})t}}\; e^{-\frac{[x-\omega'(\bar{k})t]^2}{4\left[a^2+\frac{i}{2}\omega''(\bar{k})t\right]}}$$

3. The envelope, which is obtained by looking at the absolute value (thus neglecting the oscillatory part), is (recall that $a\in\mathbb{R}$)

$$|z| = \frac{\sqrt{2\pi}}{\left(4a^4+\omega''(\bar{k})^2t^2\right)^{\frac{1}{4}}}\, e^{-\frac{a^2}{4a^4+\omega''(\bar{k})^2t^2}\cdot\left[x-\omega'(\bar{k})t\right]^2}.$$

Profile centred at $x=\omega'(\bar{k})t$ with width$^2 \sim 4a^2+\frac{\omega''(\bar{k})^2}{a^2}t^2$

By the way, this means that the amplitude of this wave decreases with time.

Ex 11 Sub in plane wave $u(x,t)=e^{i(kx-\omega t)}$:

(a) $$u_t + u_x + \alpha u_{xxx} = 0$$

$$\rightsquigarrow \; -i\omega + ik - i\alpha k^3 = 0 \quad \Rightarrow \quad \omega(k) = k - \alpha k^3$$ Dispersion relation

Phase velocity $\quad c(k) = \dfrac{\omega(k)}{k} = 1 - \alpha k^2$

Group velocity $\quad c_g(k) = \omega'(k) = 1 - 3\alpha k^2$

(b) $$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}$$

$$\rightsquigarrow \; -\omega^2 + \alpha^2 k^2 = \beta^2 k^2 \omega^2 \quad \Rightarrow \quad \omega(k) = \pm \frac{\alpha k}{\sqrt{1+\beta^2 k^2}}$$

(We might be interested only in the + sign physically, but let's not restrict here)

Phase velocity $\quad c(k) = \dfrac{\omega(k)}{k} = \pm \dfrac{\alpha}{\sqrt{1+\beta^2k^2}}$

Group velocity $\quad c_g(k) = \omega'(k) = \pm \alpha \left[\dfrac{1}{(1+\beta^2k^2)^{1/2}} - \dfrac{k}{\cancel{2}} \dfrac{\cancel{2}\beta^2 k}{(1+\beta^2k^2)^{3/2}} \right] = \pm \dfrac{\alpha}{(1+\beta^2k^2)^{3/2}}$

Ex 12 $$u_t + u_x + u_{xxx} + u_{\underbrace{x\ldots x}_{n}} = 0$$

Sub in plane wave $u(x,t) = e^{i(kx-\omega t)}$:

$$-i\omega + ik - ik^3 + i^n k^n = 0$$

$$\omega(k) = k - k^3 + i^{n-1}k^n$$

$\Rightarrow$ Plane wave $\quad u(x,t) = e^{ik[x-(1-k^2)t]}\, e^{-i^n k^n t}$

Dissipation (ω complex) if $n \in 2\mathbb{Z}$ (n even).

Physical dissipation (amplitude decays) if $n \in 4\mathbb{Z}$ (n is a multiple of 4).

Let $n = 4m$. $\quad u(x,t) = e^{ik[x-(1-k^2)t]}\, e^{-k^{4m}t}$. Exponential decay