TRAVELLING WAVE $u(x,t) = f(x - vt) \equiv f(\xi)$:

$$f'' = \hat{F}(f) \equiv -\frac{d\hat{V}(f)}{df}$$

$$\Rightarrow \quad \frac{1}{2}f'^2 + \hat{V}(f) = \hat{E} = \text{const}$$

$$\Rightarrow \quad f' = \pm\sqrt{2}\sqrt{\hat{E} - \hat{V}(f)}$$

Integrate by separation of variables:

$$I(f) := \int \frac{df}{\sqrt{\hat{E} - \hat{V}(f)}} = \pm\sqrt{2}\int d\xi = \pm\sqrt{2}\,(\xi - \xi_0) \qquad (*)$$

We can approximate the integral $I(f)$ for values of f close to f_0, where $\hat{V}(f_0) = \hat{E}$. These correspond to turning points if $\hat{F}(f_0) = -\frac{d\hat{V}}{df}(f_0) \neq 0$, or unstable equilibrium points if $\hat{F}(f_0) = -\frac{d\hat{V}}{df}(f_0) = 0$. Let's Taylor expand $\hat{V}(f)$ about f_0:

1) TURNING POINTS:

$$\hat{V}(f) = \hat{E} + \frac{d\hat{V}}{df}(f_0)\cdot(f - f_0) + O((f-f_0)^2) \approx \hat{E} - \hat{F}(f_0)\cdot(f - f_0)$$

≥ 0, therefore $\operatorname{sign}\hat{F}(f_0) = \operatorname{sign}(f - f_0) \equiv s$

Notation: omit $O(\ldots)$

$$\approx \hat{E} - \underbrace{\hat{F}(f_0)\cdot(f - f_0)}_{\substack{\geq 0 \text{ because we require} \\ \hat{V}(f) \leq \hat{E} \text{ to have } f \in \mathbb{R}}} = \hat{E} - |\hat{F}(f_0)|\cdot|f - f_0|$$

$$I(f) \approx \frac{1}{\sqrt{|\hat{F}(f_0)|}}\int \frac{df}{\sqrt{|f - f_0|}} = \pm\sqrt{\frac{2}{|\hat{F}(f_0)|}}\cdot\sqrt{|f - f_0|}$$

So $(*)$ becomes, squaring both sides,

$$f(\xi) \approx f_0 + 2s(\xi - \xi_0)^2 \qquad \text{where } s = \operatorname{sign}(f - f_0) = \operatorname{sign}\hat{F}(f_0).$$

2) "UNSTABLE EQUILIBRIUM" POINTS:

Generically $\frac{d^2\hat{V}}{df^2}(f_0) \neq 0$, so let's consider this case first.

Let $\frac{d^2\hat{V}}{df^2}(f_0) = -a^2$, where $a > 0$ wlog.

$$\hat{V}(f) \approx \hat{E} - \frac{a^2}{2}(f - f_0)^2$$

$$\Rightarrow \quad I(f) = \int \frac{df}{\sqrt{\hat{E}-\hat{V}(f)}} \approx \frac{\sqrt{2}}{|a|} \int \frac{df}{|f-f_0|} = \pm \frac{\sqrt{2}}{|a|} \log|f-f_0|$$

So

$$\frac{\cancel{\sqrt{2}}}{|a|} \log|f-f_0| = \pm \cancel{\sqrt{2}} \, (\xi - \xi_0)$$

$$\Rightarrow \quad f(\xi) = f_0 \pm e^{\pm |a|(\xi-\xi_0)} \qquad \text{(uncorrelated signs)}$$

In order for $f \longrightarrow f_0$, we need to pick the sign in the exponent as follows:

$$f(\xi) = f_0 + s\, e^{-|a|\cdot|\xi-\xi_0|}, \qquad s = \mathrm{sign}\,(f-f_0).$$

f tends to f_0 only when $\xi \rightarrow +\infty \;/ -\infty$, and it approaches that limiting value exponentially slowly.

* Work out what happens if the leading order in the Taylor expansion of $\hat{E}-\hat{V}(f)$ is the n^{th} order, e.g.

$$\hat{E}-\hat{V}(f) \approx c_n (f-f_0)^n \qquad (c_n \neq 0 \text{ is a constant})$$

with $\underline{n>2}$.