Periodic Monopoles - Dynamics and the Dual Picture

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CPT Student Seminar

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[based on RM 1212.4481 and RM & Ward 130x.xxxx]

Contents

SDYM

- Yang-Mills action

$$S = \int_{\mathbb{R}^4} \mathrm{tr}(F \wedge *F)$$

- Stationary points from Bogomolny argument $\Rightarrow$ F (anti)-self-dual

$$F = *F \qquad \Longrightarrow \qquad F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

- Independence from x^4 gives Bogomolny (monopole) equations,

$$F = *D\Phi \qquad \Longrightarrow \qquad F_{ij} = \epsilon_{ijk}\left(\partial_k\Phi + [A_k, \Phi]\right) \ldots$$

- dimensionally reduce again for Hitchin equations,

$$D_{\bar{s}}\Phi = 0 \qquad \text{and} \qquad F_{s\bar{s}} = -\tfrac{1}{4}[\Phi, \Phi^\dagger] \ldots$$

- and again for Nahm equations,

$$\partial_s A_i = \tfrac{1}{2}\,\epsilon_{ijk}\,[A_j, A_k].$$

Periodic Monopole

- $\hat{A}$, $\hat{\Phi} \in \mathfrak{su}(2)$ - i.e. 2×2 traceless anti-Hermitian matrices.

Topology

- Magnetic charge given by first Chern class,

$$k = \lim_{R\to\infty} \int_{\rho=R} \frac{\mathrm{tr}(\hat{F}\hat{\Phi})}{4\pi\|\hat{\Phi}\|}$$

Boundary conditions

- Asymptotically Abelian, $\hat{A} \sim \hat{A}_\infty \sigma_3$, $\hat{\Phi} \sim \hat{\Phi}_\infty \sigma_3$
- Resembles Dirac chain, $\nabla^2 \hat{\Phi}_\infty = 0$, $\hat{\Phi}_\infty \sim \log(\rho)$

$$(\hat{A}_z + \mathrm{i}\hat{\Phi})_\infty \sim \underbrace{\ell \log(\zeta)}_{\text{charge + s.b.}} + \underbrace{\mathfrak{v}}_{\text{size}} + \underbrace{\mathcal{O}\left(\zeta^{-1}\right)}_{\text{moduli}}$$

- For monopoles in $\mathbb{R}^3$, charge is given by *sub-leading* term - interesting implications for SU(3).

Nahm Transform

- Powerful tool to solve the Bogomolny equations, it is an adaptation of the ADHM construction for instantons. [Corrigan & Goddard '84]
- Nahm equations are easier to solve than Bogomolny - but inverse transform hard: look for approximate or numerical solutions.
- Bijection between two systems satisfying the SDYM equations.
- Roughly, swap rank of gauge group and 'soliton number'.
- The manifolds we're interested in can be related by the example of SDYM on a 4-torus, which is self-reciprocal under Nahm transform. [Braam & van Baal '89]

Nahm Transform

Arrange instantons or monopoles in a lattice. The Nahm transform allows us to consider the self-duality equations on the reciprocal lattice. [Jardim '04]

	'physical space'	'Nahm space'
instanton	$\mathbb{R}^4$	$\bullet$
caloron (periodic instanton)	$\mathbb{R}^3 \times \hat{S}^1$	S^1
doubly periodic instanton	$\mathbb{R}^2 \times \hat{T}^2$	T^2
monopole	$\mathbb{R}^3$	$\mathbb{R}$
periodic monopole	$\mathbb{R}^2 \times \hat{S}^1$	$\mathbb{R} \times S^1$
doubly periodic monopole	$\mathbb{R} \times \hat{T}^2$	$\mathbb{R} \times T^2$

(N.B. the Nahm transform has only been proved for some of these!)

Inverse Nahm for Periodic Monopole

For an SU(2) charge k periodic monopole, solve rank k Hitchin equations on $\mathbb{R} \times S^1$,

$$F_{s\bar{s}} \,=\, -\tfrac{1}{4}[\Phi, \Phi^\dagger] \qquad\qquad D_{\bar{s}}\Phi \,=\, \partial_{\bar{s}}\Phi + [A_{\bar{s}}, \Phi] \,=\, 0,$$

where $\det(\Phi)$ is given by the spectral curve.

Solve the equation

$$\Delta\Psi \,=\, \begin{pmatrix} \mathbf{1}_k \otimes (2\partial_{\bar{s}} - z) + 2A_{\bar{s}} & \mathbf{1}_k \otimes \zeta - \Phi \\ \mathbf{1}_k \otimes \bar{\zeta} - \Phi^\dagger & \mathbf{1}_k \otimes (2\partial_s + z) + 2A_s \end{pmatrix} \Psi \,=\, 0.$$

Construct the monopole fields from normalised solutions $\iint \Psi^\dagger\Psi = \mathbf{1}_2$,

$$\hat{\Phi} \,=\, \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}r \int_{-\pi/\beta}^{\pi/\beta} \mathrm{d}t\,(r\Psi^\dagger\Psi) \qquad\qquad \hat{A}_i \,=\, \int_{-\infty}^{\infty} \mathrm{d}r \int_{-\pi/\beta}^{\pi/\beta} \mathrm{d}t\,(\Psi^\dagger\partial_i\Psi).$$

Gauge transformations act as $\Psi \mapsto U(s)^{-1}\,\Psi\,\hat{g}(\zeta, z)$ with $U = h \otimes g$.

Spectral Curve

As for the monopole in $\mathbb{R}^3$, it is useful to consider solutions of

$$(\partial_z + \hat{A}_z + \mathrm{i}\hat{\Phi})V(\zeta, z) \,=\, 0 \qquad \text{with} \qquad V(\zeta, 0) \,=\, \mathbf{1}_2.$$

$V(\zeta, \beta)$ defines the holonomy. Its characteristic equation is a polynomial in $w = \mathrm{e}^{\beta s}$ [Cherkis & Kapustin '01, '03]

$$w^2 + P_k(\zeta)w + 1 \,=\, 0.$$

This defines a curve in $\mathbb{C} \times \mathbb{C}^*$. The same curve can be written

$$\zeta^k - \mathrm{tr}(\Phi)\zeta^{k-1} + \ldots + (-1)^k \det(\Phi) \,=\, 0.$$

$P_k(\zeta)$ has $2k + 2$ coefficients: 2 parameters (b.c.s), 2 for centre of mass, and $2k - 2$ moduli. This is half the number of moduli for a charge k monopole in $\mathbb{R}^3$.

Charge 1

The spectral curves are

$$w^2 - 2\zeta w/C + 1 = 0 \qquad\qquad \zeta = \det(\Phi)$$

Hitchin data is smooth and Abelian [Ward '05]

$$A = 0 \qquad\qquad \Phi = C\cosh(\beta s).$$

The Nahm equation

$$\Delta\Psi = \begin{pmatrix} 2\partial_{\bar{s}} - z & \zeta - \Phi \\ \bar{\zeta} - \Phi^\dagger & 2\partial_s + z \end{pmatrix} \Psi = 0$$

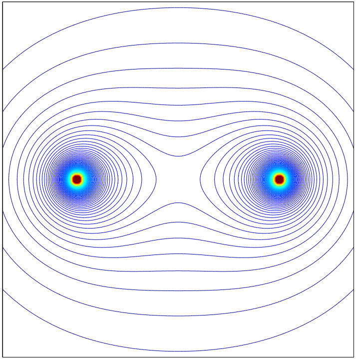
has two distinct solutions if ζ remains away from $\zeta = \pm C$. In this region, the approximate monopole fields are

$$\hat{A}_\zeta = \hat{A}_{\bar{\zeta}} \approx 0 \qquad\qquad -(\hat{A}_z + \mathrm{i}\hat{\Phi}) \approx \frac{1}{\beta}\cosh^{-1}\left(\frac{\zeta}{C}\right)\sigma_3$$

$\hat{A}_\zeta$ has off-diagonal terms which decay exponentially away from $\zeta = \pm C$. Note $\hat{\Phi}$ can be read off by solving the 'w' spectral curve for $s(\zeta)$.

Charge 1

Energy density is given by $\mathcal{E} = \nabla^2 \|\hat{\Phi}\|^2$. On a cross section of the chain,



- Energy peaks are located at $\zeta = \pm C$.
- Total energy diverges as $\log(\rho)$, but can still consider relative moduli space. [Cherkis & Kapustin '02]

Spectral Approximation

Results suggest a general pattern:

- We assume monopole fields can be read off from the spectral curve.
- Peaks in energy density are then found at the values of ζ where the eigenvalues of $V(\zeta, \beta)$ coincide.
- There are $2k$ such *spectral points*, which come in pairs.
- This provides a way of studying higher charge chains, or larger gauge groups, simply from the spectral curves.
- Approximation improves for monopole size $C \gg$ period β, i.e. in the limit of z independence.

Charge 2 - Spectral Approximation

The spectral curves are

$$w^2 - (2\zeta^2 - K)w/C + 1 = 0 \qquad \zeta^2 = -\det(\Phi)$$

with spectral points at

$$\zeta = \pm\sqrt{K/2 \pm C}.$$

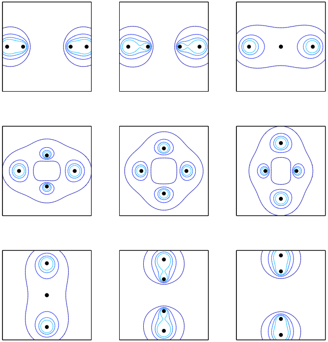
K is a complex modulus.

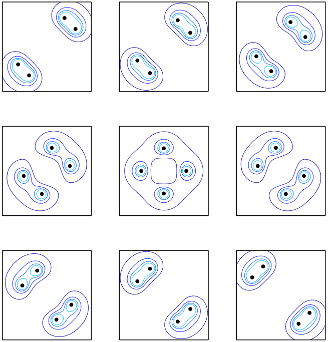
Impose symmetries on the spectral curve, e.g. $(w, \zeta; K) \mapsto (\bar{w}, \bar{\zeta}; \bar{K})$ and $(w, \zeta; K) \mapsto (-w, \mathrm{i}\zeta; -K)$ show that $K \in \mathbb{R}$ is a one-parameter family where the spectral points undergo right angled scattering.

Similarly $(w, \zeta; K) \mapsto (-\mathrm{i}\bar{w}, \mathrm{e}^{\mathrm{i}\pi/4}\bar{\zeta}; \mathrm{i}\bar{K})$ shows that $K \in \mathrm{i}\mathbb{R}$ is another one-parameter family.

Note $K = 0$ has enhanced symmetry, while if $K = \pm 2C$ two spectral points coincide.

Charge 2 - Spectral Approximation





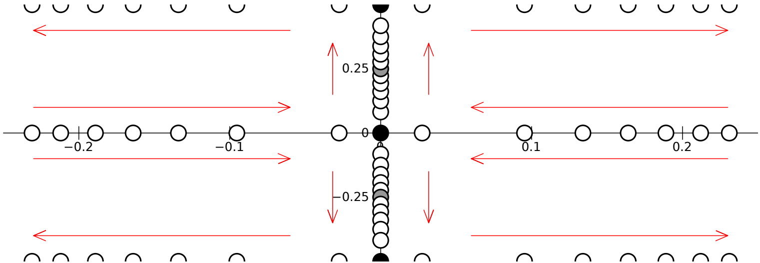
Charge 2 - Zeroes on the Cylinder

The Hitchin field has

$$-\det(\Phi) \;=\; C\cosh(\beta s) + K/2.$$

The two zeroes of this function also undergo scattering as K is varied:

$K \in \mathbb{R}$



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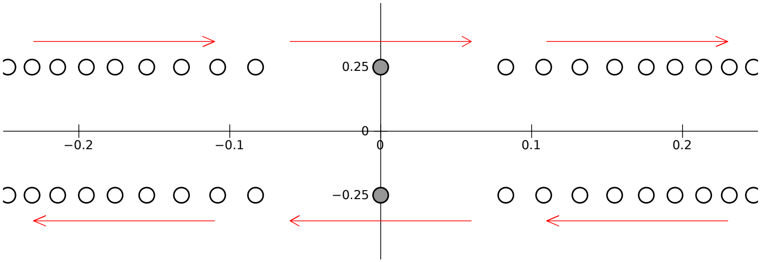


Charge 2 - Moduli Space

The monopole field $\hat{\phi} = \hat{A}_z + \mathrm{i}\hat{\Phi}$ is known explicitly in the spectral limit.

A metric on the moduli space can be obtained by varying the fields with respect to the modulus K,

$$g \sim \dot{K}\dot{\bar{K}} \int_{\mathbb{R}^2} \mathrm{tr}\left(\partial_K \hat{\phi}\, \partial_{\bar{K}} \hat{\phi}^\dagger\right) \rho\, \mathrm{d}\rho\, \mathrm{d}\theta$$

The gauge condition that variations are orthogonal to gauge orbits is automatically satisfied (dim. red. of $D_\mu(\partial_K \hat{A}_\mu) = 0$).

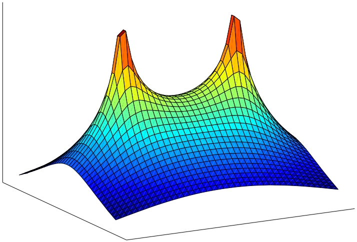
In terms of the spectral points $\zeta_i = \pm\sqrt{K/2 \pm C}$,

$$g \sim \dot{K}\dot{\bar{K}} \int_{\mathbb{R}^2} \frac{1}{|\zeta - \zeta_1||\zeta - \zeta_2||\zeta - \zeta_3||\zeta - \zeta_4|}\, \rho\, \mathrm{d}\rho\, \mathrm{d}\theta$$

Perform integral for $K = 0, \pm 2C$, otherwise numerically...

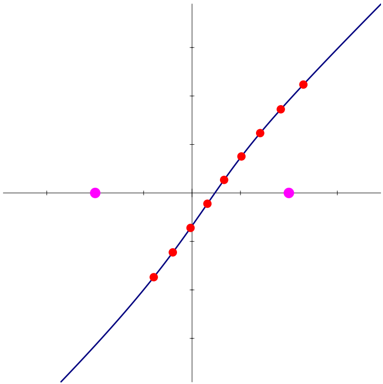
Charge 2 - Moduli Space

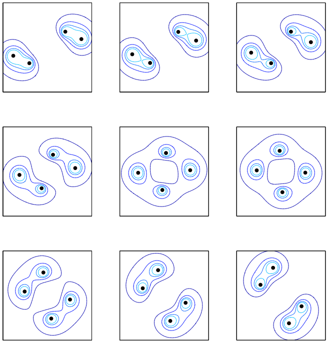
Conformal factor as function of K:



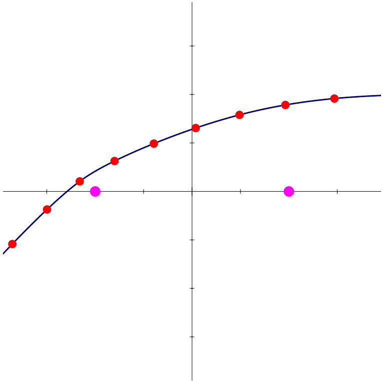
- $K = 0$ and $K = \pm 2C$ are special!
- $K \in \mathbb{R}$ and $K \in \mathrm{i}\mathbb{R}$ are indeed geodesics.
- Otherwise, numerically evolve geodesic equations...

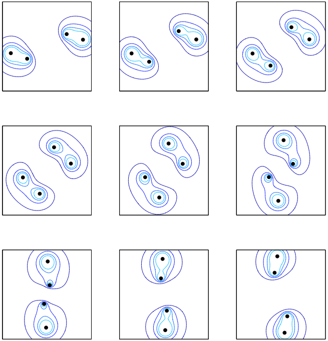
Charge 2 - Moduli Space





Charge 2 - Moduli Space





Note different constituent behaviour according to whether or not the geodesic crosses the line segment $-2C \leq K \leq 2C$.

Charge 2 - Nahm Transform

Recall we are to solve rank 2 Hitchin equations

$$F_{s\bar{s}} = -\tfrac{1}{4}[\Phi, \Phi^\dagger] \qquad D_{\bar{s}}\Phi = \partial_{\bar{s}}\Phi + [A_{\bar{s}}, \Phi] = 0,$$

with $-\det(\Phi) = C\cosh(\beta s) + K/2$.

Up to a gauge, we can write [Harland & Ward '09]

$$\Phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix} \qquad A_{\bar{s}} = a\sigma_3 + \alpha\Phi$$

α encodes the other moduli (z-offset and relative phase).

Now, look at symmetries of the Hitchin equations and the Nahm operator. Imposing invariance under $(\zeta, z) \mapsto (\zeta, -z)$ shows $\alpha = 0$ is a geodesic submanifold. Justifies looking for geodesics on K plane!

Charge 2 - Nahm Transform

Ansatz for Hitchin fields:

$$\Phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix} \qquad \Rightarrow \qquad \mu_+\mu_- = C\cosh(\beta s) + K/2$$

Note $\det(\Phi)$ has two zeroes. This gives two distinct smooth solutions for Φ according to their allocation between the non-zero components.

- 'zeroes together' [Harland]

$$\mu_+ = C\cosh(\beta s) + K/2 \qquad \mu_- = 1$$

- 'zeroes apart' [Harland & Ward '09]

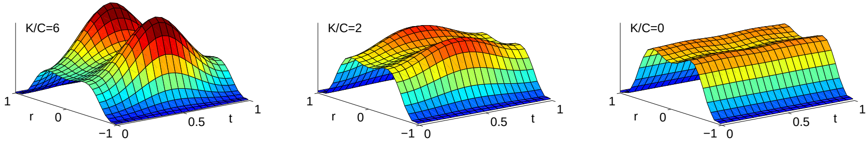
$$\mu_\pm = \sqrt{C/2}\left(e^{\beta s/2} + \lambda^{\pm 1} e^{-\beta s/2}\right) \qquad 2C\lambda^{\pm 1} = K \pm \sqrt{K^2 - 4C^2}$$

Note that both solutions have the same spectral limit.

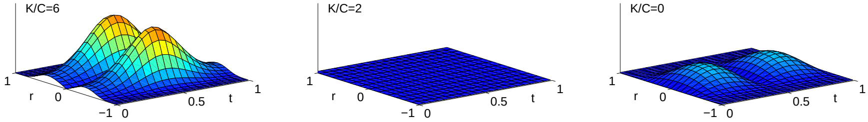
Charge 2 - Lumps on Cylinder

On the $\alpha = 0$ geodesic, solve Hitchin equations numerically. Plot $|F|$,

- 'zeroes together'



- 'zeroes apart'



size/period ratio now determined by $1/C$

Charge 2 - Limits of C

$C \gg 1$

- monopole size $\gg$ period (z independence)
- spectral approximation holds
- sharply localised lumps on cylinder
- lumps closely track zeroes of $\det(\Phi)$

$C \ll 1$

- chain of small monopoles
- wide peaks on cylinder, approach Nahm data (singularities develop at finite r)

$\alpha \neq 0$

- z offset $\rightarrow$ t-holonomy in central region
- relative phase $\rightarrow$ relative phase on lumps

Charge 2 - Metric from Cylinder - $C \gg 1$

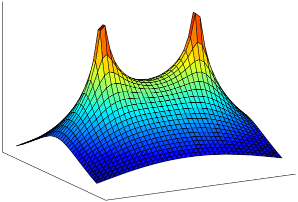
- Simple solution to Hitchin equations.
- Gauge condition can be solved explicitly, $\Phi \mapsto \Phi' = \sqrt{\det(\Phi)}\,\sigma_1$.
- Metric depends only on $\det(\Phi)$

  $\Rightarrow$ same for 'zeroes together' and 'zeroes apart'

$$g \sim \dot{K}\dot{\bar{K}} \int_{\mathbb{R}\times S^1} \frac{|\partial_K \det(\Phi')|^2}{|\det(\Phi')|} \sim \dot{K}\dot{\bar{K}} \int_{\mathbb{R}\times S^1} \frac{\mathrm{d}r\,\mathrm{d}t}{|C\cosh(\beta s) + K/2|}$$

- Conformal factor agrees with spectral approximation!
- Asymptotically $\log(K)/K$

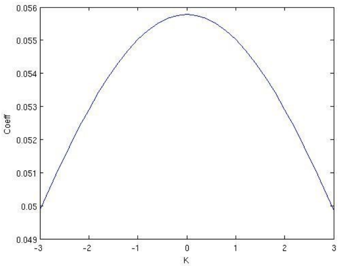
  [agreement with Cherkis & Kapustin '02]
- peaks $\sim \log(|K \pm 2C|)$



Charge 2 - Metric from Cylinder - $C \ll 1$

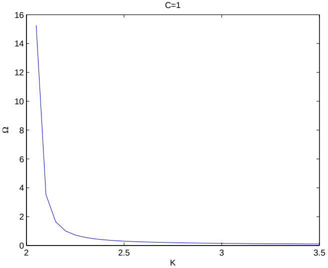
- So far, only numerically.
- Approach rotational symmetry.
- Asymptotically $\log(K)/K$.

'zeroes together'



smooth
scatter in plane
A-H cone?

'zeroes apart'



peak $\sim (K - 2C)^{-1}$
double scattering along z
A-H trumpet?

Summary & Outlook

- Bogomolny eqs on $\mathbb{R}^2 \times \hat{S}^1 \xrightarrow{\text{Nahm transform}}$ Hitchin eqs on $\mathbb{R} \times S^1$.
- Chain of large monopoles $\rightarrow$ z independence $\rightarrow$ all information contained in holonomy $\rightarrow$ *spectral approximation*.
- Nahm transform $\rightarrow$ motion of lumps on dual space.
- Moduli space: two solutions which coincide in spectral limit.
- Spectral approximation can be applied to e.g. charge 3 or SU(3).
- Can the constituents be studied in their own right?

END