Instantons, Hyperbolic Monopoles, and the Nahm Transform

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Outline

Self-Dual Yang-Mills Equations

The pure $SU(2)$ Yang-Mills action for a gauge field 1-form A:

$$S = -\int_{\mathbf{R}^4} \mathrm{Tr}(F \wedge *F)$$

leads via a Bogomolny argument

$$S = -\int_{\mathbf{R}^4} \mathrm{Tr}(F \wedge F) - \frac{1}{2}\int_{\mathbf{R}^4} \mathrm{Tr}((F - *F) \wedge *(F - *F))$$

to SDYM:

$F = *F$

Finite action solutions to these are called 'instantons'. The instanton charge is given by:

$$N = -\frac{1}{8\pi^2}\int_{\mathbf{R}^4} \mathrm{Tr}(F \wedge F)$$

SDYM is a 'master equation' for integrable systems.

The ADHM construction

SDYM are hard, non-linear PDEs. Fortunately, we have the ADHM construction, which gives *all* instanton solutions in terms of quaternionic matrices satisfying algebraic constraints. The data for an $SU(2)$ N-instanton is:

$$\widehat{M} = \begin{pmatrix} L \\ M \end{pmatrix}$$

where L is a row of N quaternions, and M is a symmetric $N \times N$ matrix of quaternions. Let $x = x_1 + x_2 i + x_3 j + x_4 k$ be an arbitrary quaternion. Then define:

$$\Delta(x) = \widehat{M} - x \begin{pmatrix} 0 \\ 1_N \end{pmatrix}$$

The data must satisfy, for all x:

$$\Delta(x)^\dagger \Delta(x) = R_0(x)$$

where $R_0(x)$ is a non-singular real matrix.

The ADHM Construction (cont.)

To construct the gauge field from the ADHM data, one must find a unit norm $(N+1)$-component column vector $\Psi(x)$ satisfying

$$\Psi^\dagger \Delta(x) = 0$$

The instanton gauge field is then given by:

$$A_\mu = \Psi^\dagger \partial_\mu \Psi$$

BPS Magnetic Monopoles

Let's dimensionally reduce the pure Yang-Mills action down to 3 dimensions by setting all the fields to be independent of x_4, and set $\Phi = A_4$:

$$E = -\frac{1}{2}\int_{\mathbf{R}^3} \mathrm{Tr}(F \wedge *F) + \mathrm{Tr}(D\Phi \wedge *D\Phi)$$

The SDYM equations become:

$$*F = D\Phi$$

One imposes that $|\Phi| \to 1$ at spatial infinity, defining a map $\Phi_\infty : S^2_\infty \to S^2$ whose winding number is the charge of the monopole. Finite energy solutions to this equation are called 'BPS magnetic monopoles'.

The Nahm transform

The monopole version of the ADHM construction is called the 'Nahm transform'. Every N-monopole is equivalent to a solution of the 'Nahm equations':

$$\frac{dT_i}{ds} = \frac{1}{2} i\epsilon_{ijk}[T_j, T_k]$$

where $T_i, i = 1, 2, 3$ are $N \times N$ Hermitian matrix functions of $s \in [-1, 1]$. Although slightly more tractable, these equations are also very hard to solve and even if one has Nahm data the inverse Nahm transform must usually be performed numerically.

The Ercolani-Sinha solution

There is only one known infinite family of monopoles for every value of the topological charge N [Ercolani & Sinha '89]. These solutions are based on the N-dimensional irreducible representations $\{J_i\}_{i=1,2,3}$ of the Lie algebra $\mathfrak{su}(2)$, with commutation relations $[J_i, J_j] = i\epsilon_{ijk} J_k$. Plugging $T_i = -f_i(s) J_i$ into the Nahm equations gives the spinning top equations:

$$\frac{df_1}{ds} = f_2 f_3; \quad \frac{df_2}{ds} = f_3 f_1; \quad \frac{df_3}{ds} = f_1 f_2$$

These can be solved exactly in terms of elliptic functions. The solutions correspond to N equally spaced monopoles strung out along a line. The solution has axial symmetry for a special value of the elliptic parameter.

The Spectral Curve

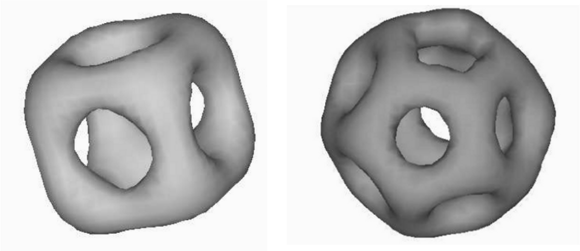
Monopoles were originally shown to be equivalent to Nahm data via a third description: the spectral curve. The spectral curve is a Riemann surface encoding the monopole. It lives in the tangent bundle of S^2, which is the same as the space of oriented lines in $\mathbf{R}^3$. To define the spectral curve, take an oriented line γ and consider the Hitchin equation:

$$(D_\gamma - i\phi)v = 0$$

where D_γ is the covariant derivative along γ. The spectral curve is the set of lines on which this equation has a normalisable solutions.

Pretty Pictures of Monopoles

Most of the few known examples of monopoles have been found by imposing Platonic symmetries on the Nahm data.



A cubic and a dodecahedral monopole, from Houghton and Sutcliffe CMP 180 (1996) 343-362, and Nonlinearity 9, 385

Hyperbolic Monopoles

One can generalise the Bogomolny equation to 3-manifolds with an arbitrary metric h_{ij}:

$$\frac{1}{2}\sqrt{\det h}\,\epsilon_{ijk}h^{jl}h^{km}F_{lm} = D_i\Phi$$

In the Euclidean case $h_{ij} = \delta_{ij}$. We will be interested in monopoles on hyperbolic space, which are often simpler in some ways than their Euclidean counterparts.

The two most commonly used models of hyperbolic space are the half-space and ball models.

$\mathbf{R}^4 \setminus \mathbf{R}^2$ is conformal to $\mathbf{H}^3 \times S^1$

Let $\mathbf{R}^4$ have standard Cartesian coordinates x_1, x_2, x_3, x_4, and Euclidean metric:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

Now define $x_3 + ix_4 = re^{i\chi}$. The metric becomes:

$$ds^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\chi^2$$

If we remove the 2-plane $r = 0$, this metric is conformal to:

$$ds^2 = \frac{1}{r^2}(dx_1^2 + dx_2^2 + dr^2) + d\chi^2$$

which is the metric on $\mathbf{H}^3 \times S^1$. The removed plane is the 'boundary' of hyperbolic space.

Circle-Invariant Instantons are Hyperbolic Monopoles

Suppose we have an instanton which is invariant under the standard rotation of χ. Since $\mathbf{R}^4 \setminus \mathbf{R}^2$ is conformally equivalent to $\mathbf{H}^3 \times S^1$, and the SDYM equations are conformally invariant, we can dimensionally reduce along the circle direction to give a hyperbolic monopole. In this case the asymptotic magnitude of the Higgs field must be a half-integer p. The instanton charge I and the monopole charge N are related by:

$$I = 2pN$$

(One can take this further and say that since a Euclidean instanton is equivalent to a hyperbolic periodic instanton, which is equivalent to a hyperbolic monopole with a loop group as its gauge group, there is a sense in which 'instantons are monopoles' [Garland & Murray '89]).

The Braam-Austin construction

Interestingly, one can write the ADHM data for circle-invariant instantons in terms of a set of difference equations defined on a lattice with $2p$ points, indexed by j [Braam & Austin '90]. If A_j, B_j, C_j and D_j are $N \times N$ matrices, then the equations are:

$$A_{j+1} = D_j A_j D_j^{-1}$$
$$B_{j+1} = C_j^{-1} B_j C_j$$
$$C_{j+1} D_{j+1} = D_j C_j + [A_{j+1}, B_{j+1}]$$

These are a discretisation of the Nahm equations. If we set $C = (2h)^{-1} I + \frac{1}{2} i T_3 = D$ where h is the lattice spacing, and $B = \frac{1}{2}(T_1 + i T_2) = -A^\dagger$, then let $h \to 0$, we get back the Nahm equations.

Problems with Braam-Austin

- The Braam-Austin equations have difficult boundary conditions.
- Braam-Austin uses a very abstract version of ADHM, involving holomorphic bundles over twistor space etc., rather than simple quaternionic matrices.
- No-one has solved the Braam-Austin equations beyond $N = 1$.
- It's unclear if Braam-Austin provides any simplification at all for low p.
- Braam-Austin is only adapted to the half-plane model of hyperbolic space.

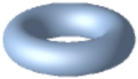
Platonic Hyperbolic Monopoles

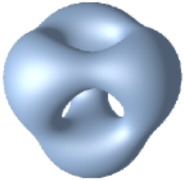
[Manton & Sutcliffe '12] gives a list of conditions which are sufficient for an instanton to be circle invariant. These conditions are adapted to the ball model of $\mathbf{H^3}$, which allows one to investigate compatible symmetry subgroups of $SO(3)$. The conditions for ADHM data $\widehat{M} = \begin{pmatrix} L \\ M \end{pmatrix}$ are:

- M is pure quaternion and symmetric,
- $\widehat{M}^\dagger \widehat{M} = 1_N$,
- $LM = \mu L$, where μ is a pure quaternion, and L is non-vanishing.

Unfortunately these only give hyperbolic monopoles with $p = 1/2$.

More Pretty Pictures





Pictures from Manton and Sutcliffe, arXiv:1207.2636

An axially symmetric monopole, with ADHM data

$$\widehat{M} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2}k \\ i & j \\ j & -i \end{pmatrix}$$

A tetrahedrally symmetric monopole, with ADHM data

$$\widehat{M} = \frac{1}{\sqrt{3}}\begin{pmatrix} i & j & k \\ 0 & k & j \\ k & 0 & i \\ j & i & 0 \end{pmatrix}$$

Solutions to discrete Nahm equations

- In [Ward '90], Ward shows how the continuous Nahm equations can be reduced to Toda lattice equations. Ward also shows in [Ward '98] that the same is true for discrete Nahm equations.
- Ward then specialises to the charge 2 case and solves the resulting discrete Toda equations in terms of elliptic functions.

Ward's $N = 2$ Solution

If we make the ansatz

$$C = \mathrm{diag}(h^{-1}\sqrt{1+2hw}, h^{-1}\sqrt{1-2hw}) = D$$
$$B = \begin{pmatrix} 0 & u-v \\ u+v & 0 \end{pmatrix} = -A^T$$

where u, v, w are real-valued lattice functions, then the discrete Nahm equations become

$$w_+ = w + 2hu_+v_+$$
$$u_+ = (u - 2hwv)/\sqrt{1-4h^2w^2}$$
$$v_+ = (v - 2hwu)/\sqrt{1-4h^2w^2}$$

which are a discrete-time version of the spinning top equations.

Ward's $N = 2$ Solution (cont.)

- Although it seems likely, it is *not* clear that these solutions correspond to hyperbolic monopoles. We need to understand the Braam-Austin construction better before we can claim this.
- Note that $B = u\sigma_1 - iv\sigma_2$, where $\{\sigma_i\}_{i=1,2,3}$ are the Pauli matrices. This suggests that Ward's solution should generalise to all N, giving a hyperbolic version of the Ercolani-Sinha family of monopoles.

Outlook

- We would like to make the Braam-Austin construction more intelligible and give explicit solutions.
- Hopefully one can use Braam-Austin to construct Platonic hyperbolic monopoles with $p > 1/2$.
- This work was partly motivated by a desire to understand magnetic bags in hyperbolic space. It would be useful to find examples of hyperbolic monopoles which allow us to take the large charge limit.
- We would like to find some of the moduli space metrics for hyperbolic monopoles.

Thank You