Some aspects of dynamics of topological solitons*

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We present a general review of the dynamics of topological solitons and then discuss some recent work on the scattering of various solitons (in 1 and 2 dimensions) on potential obstructions.

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I. INTRODUCTION

Topological solitons arise in various areas of applied mathematics and mathematical description of some processes in physics[1].

In many applications of mathematics to the description of physical processes one uses either point like objects or plane waves. Topological solitons are different. They describe object that are localised in space (but not localised to a point). So if one looks at their energy density one finds that it is described by a function which is essentially nonzero in a finite region; *ie* it is significantly nonzero in a small region and goes to zero, exponentially or as an inverse power, as one moves away from this region.

Their stability is guaranteed by topological considerations, most often associated with the topology of $S^N \rightarrow S^N$ maps.

The simplest example of such maps involves the Sine-Gordon kinks (in 1+1 dimensions). In this case $N = 1$. As is well known the Lagrangian density is given by

$$\mathcal{L} = \partial_\mu \phi \cdot \partial^\mu \phi \,-\, \lambda \sin(\phi)$$

Solutions of the equations of motion which follow from from I are well know. They involve kinks and antikinks, which are topological solitons; breathers, which can be thought of as interesting bound states of kinks and antikinks, and many other solutions, less interesting from our point of view.

In two spatial dimensions (now $N = 2$) the solitons are based on sigma models. The Lagrangian in this case is given by:

$$\mathcal{L} = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \theta_S \left[(\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi})^2 - (\partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi})(\partial^\mu \vec{\phi} \cdot \partial^\nu \vec{\phi}) \right] - V(\vec{\phi}) \tag{1}$$

where

$$V(\vec{\phi}) = \mu(1 - \phi_3^2) \tag{2}$$

and where the vector $\vec{\phi}$ lies on the unit sphere $\mathcal{S}^2$ hence $\vec{\phi} \cdot \vec{\phi} = 1$. To have a finite potential energy the field at spatial infinity is required to go to $\phi_3 = \pm 1$, $\phi_1 = \phi_2 = 0$. In this work we choose "the vacuum" to be defined as $\phi_3 = +1$.

The three terms in 1 are, from left to right, he pure $\mathcal{S}^2$ sigma model, the Skyrme and the potential term. The last two terms are needed to stabilize the solitons. They have no influence on the topology - which is still based on the

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topology of $S^2 \to S^2$ maps as imposing the spatial infinity boundary condition has defined a one-point compactification of R_2, allowing us to consider $\vec{\phi}$ on the extended plane $R_2 \bigcup \infty$ topologically equivalent to $\mathcal{S}^2$.

In 3 spatial dimensions we have skyrmions and monopoles but given the shortage of time they will not be discussed here; the interested reader is sent to [1].

II. DYNAMICS

The dynamics of the Sine Gordon kinks is well known, so we will say very little about it here. The kinks reflect from each other and, really, not much can happen as the motion is in one dimension.

For the two dimensional sigma model solitons we have two basic possilities of the dynamics. The relativistic dynamics (based on the lagrangian above) and the norelativistic one - corresponding to the Landau-Lifshitz equation.

In the later case the equation of motion is given by

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{\phi} \times \frac{\partial L}{\partial \vec{\phi}}$$

where for L we now take the spatial part of $\mathcal{L}$, *ie* of (1).

The dynamics of both cases is very different.

In the relativisitic case we have the familiar 90^0 scattering. Thus when two 2-dimensional solitons are sent towards each other head on, the system evolves in such a way that after the scattering the two outgoing solitons are moving in the direction at 90^0 with respect to their motion towards each other.

This has been explained in many ways; the most compelling involves the indistinguishability of solitons ([1]). As the system of two solitons is described by a function which is symmetric with respect of the interchange of their positions this is built into their phase space which, in terms of the relative position is really described by R^2 mod a reflection in the line joining their positions. Hence, effectively, the space is $\frac{R^2}{Z}$, where Z describes this reflection, and so is a cone. A straight line motion on this space, going through the vertex of the cone, is described by a 90^0 motion when viewed in R^2.

In the nonrelativistic case the situation is completely different.

First of all the equations involve only first order time derivatives and so the motion takes place in a lower dimensional phase space.

This has been analysed in great detail by Papanicolaou and Tamaras [2] who showed that when we have a system of two bubbles (they are described by 2 dimensional solitons) one can introduce $\vec{r} = (x1, x2)$ - a 2 dimensional vector describing their relative position and the relative momentum $\vec{p}$. However the momentum satisfies

$$p_i \sim \alpha\, \epsilon_{ij} x_j$$

and so the equation motion is of the form

$$\frac{d^2 x_i}{dt^2} \sim \alpha \epsilon_{ij} \frac{dx_j}{dt}$$

resulting in a motion around a circle.

In 3 spatial dimensions the dynamics of solitons is even more complicated - but, interestingly, some aspects of it can be related to the dynamics in 2 dimensions (through projections into various planes).

All this discussion concerned solitons moving in free space, *ie* in space with no potential obstructions. In the next section we look at case of the scattering of solitons when we do have an obstruction - either in the shape of a potential bump or a potential hole.

III. POTENTIAL OBSTRUCTION

There are various ways of introducing a potential hole or a potential barrier. However, given that the soliton field, strictly speaking, is never zero, even though it vanishes exponentially as we move away from its position, this potential has to be introduced in such a way that it does not change the "tail" of the soliton *i.e.* it has to vanish when, in one dimension $\phi = 0$ or π and in two dimensions when $\phi_3 = 1$.

A. Sigma Models

A possible way to introduce such an obstruction is to add an extra term to the Lagrangian which vanishes when $\phi_3 = 1$. Of course, there are many possible choices of such terms but given that our Lagrangian already contains a term with such a property we exploit this fact and choose to add $\alpha(1 - \phi_3^2)$ in some region of x and y. We choose this term to be independent of y so that the obstruction on the potential energy landscape, located in some finite region of x, say at positive x, resembles a trough in the "hole" case or a dam in the "barrier" case. Then sending the soliton from a point well away from this obstruction, *i.e.* initially placed at some sufficiently negative x, in the positive x direction, we can study the effects of the obstruction.

In our numerical simulations we have chosen the obstruction to be constant in a small range of x; this effectively corresponds to taking μ in the original Lagrangian to be given by μ_0 for x in the range of the obstruction and μ_1 elsewhere.

The case when $\mu_0 > \mu_1$ corresponds to a barrier (dam), and when $\mu_0 < \mu_1$ we have a hole (trough).

We have performed many numerical simulations of such systems, varying both the sign and value of $\mu_0 - \mu_1$ and the velocity of the incoming soliton. we have found that when we have a barrier the scattering is very elastic with the system essentially converting all its kinetic energy into the potential energy to 'climb the potential hill' and then releasing it back as a kinetic energy of either the reflected or transmitted soliton. Hence the velocity of the outgoing soliton was very close, in magnitude, to the original velocity.

For the hole the situation was very different. Varying the incoming velocities we saw many trajectories. Sometimes the solitons were transmitted, sometimes they were trapped sometimes they were reflected. In fig 1. we present plots of position in x as a function of times seen in various simulations (*ie* started with different values of initial velocities).

We see that in addition to the transmission the soliton can get trapped in the hole This trapping can lead to being permanently stuck in the hole (with the soliton radiating its excess of energy) or after a bounce or two in the hole the soliton can be ejected either forwards or backwards so that the whole process looks like a transmission or reflection! We have looked at this process in great detail [3]. There we argued that the interaction proceeds through the excitement of the vibrational modes of the soliton. Hence we constructed a simple model of these modes and their interaction with the soliton - we shall discuss this later on. Before we look at other models and in particular at the Sine Gordon model in one dimension. This model has no genuine vibrational modes so may expect the behaviour of its solitons to be somewhat different. We discuss this in the next section.

B. Sine Gordon Model

In the sine-Gordon case we can introduce the potential in two different ways - either by making the λ in (I) position dependent[4] or by altering the basic metric [5]. Here we discuss the results reported in [4]; the results of [5] are qualitatively very similar.

We have looked at the Sine Gordon model and have found that in this model, as in the sigma model in two

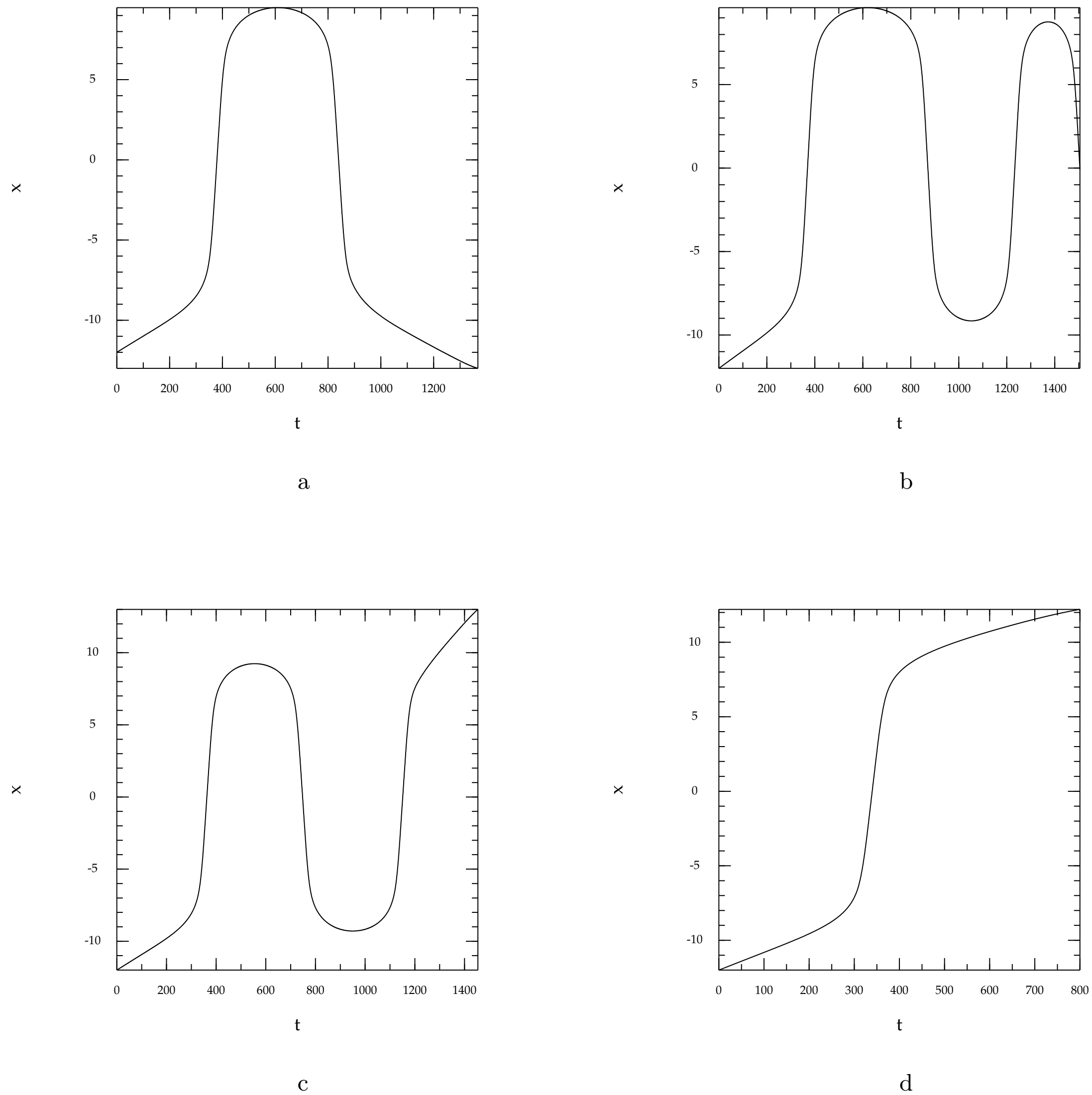

FIG. 1: Trajectory of a soliton during the scattering for a well of width $L = 10$ and depth $a = -0.2$ for $\lambda = 0.5$ and a) $v = 0.0102$, b) $v = 0.0106$, c) $v = 0.0109$, d) $v = 0.012$.

dimensions, the solitons can get trapped, be transmitted and bounce back. The process is inelastic and depends on the initial condition. It also depends on the size and the depth of the hole. If the initial condition of the soliton corresponds to an exact sine-Gordon kink moving with a given velocity then there is a well defined critical velocity above which the soliton get transmitted (with a certain loss of energy). Below this critical velocity the soliton get trapped or is reflected. The ranges of velocities, at which the soliton is reflected are very narrow. As the hole becomes shallower the critical velocity decreases and as the hole becomes narrower the number of the velocity windows at which we observe reflections gets larger, although we have not performed enough simulations to quantify this statement.

In fig.2 we present a plot of outgoing velocity as a function of the incoming velocity in the case of a relatively narrow well (λ is changed in the well from 1 down to 0.8. The hole is relatively narrow - ie a soliton fits in it about 3 times.

We note that the windows of velocities, at which we have reflections are very narrow and that there are several

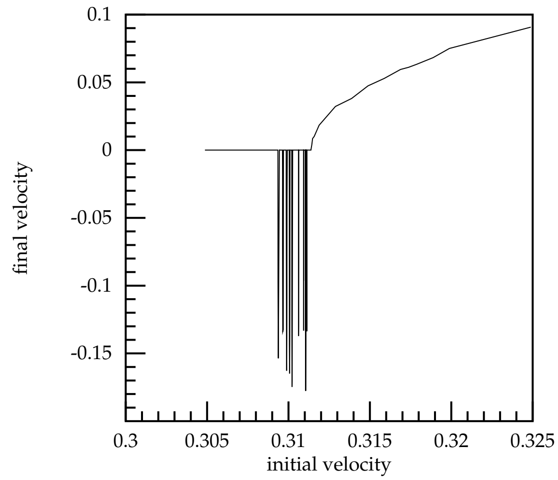

FIG. 2: Outgoing velocity of the kink as a function of its initial velocity

of them. Our results show that even in a model in which a soliton has no vibrational modes the reflections do take place. On the other hand, although the model has no vibrational modes it can radiate (*ie* we have pseudovibrational modes). An example of such a mode ia a mode which describes the variation of the slope of the kink. The usual kink solution is given by

$$\varphi(x) \;=\; 2\tan^{-1}(\exp(\theta(x-x_0)), \tag{3}$$

where x_0 is the kink's position and θ is its slope. For (3) to be a solution of the equations of motion which follow from (I) we need to set $\theta = \lambda$. However, if we put θ different from λ we excite the mode which corresponds to the variation of θ. In fact, when the kink enters the hole, where λ is different it automatically tries to adjust its slope and so it excites this mode. Of course, when this mode of the kink is excited - the kink begin to radiate and its is the interaction of this radiation with the kink itself which is reponsible for the final outcome of the scattering process.

IV. SIMPLE MODELS

Here we present simple models which partially explain what has been seen in full numerical simulations. First we consider the sigma model in 2 dimensions. In this case the soliton possesses many vibrational modes [6]. To proceed further we present a model which treats a soliton as a system of four masses (connected to each other by springs).

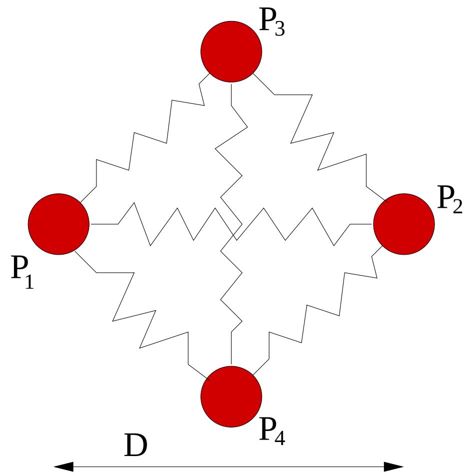

FIG. 3: Our model of vibrational modes of a sigma model soliton.

A. Four Mass model[3]

Our effective model of lowest vibrational modes of sigma model solitons involves 4 masses coupled together by strings as shown in fig. 3. The system is then sent towards the hole and is it falls in the masses begin to oscillate. These oscillations then ‘ape’ the soliton vibrations. So the energy is transferred to these oscillations and if the energy of the centre of mass is too low the system is trapped in the hole. Sometimes, when the system reaches one of the edges of the hole the energy of the oscillations gets transferred back to the system as whole (the energy of its centre of mass) and the system can come out. Whether this happens or not depends on the flow of the energy between the variational modes and the centre of energy modes. In fact, as we have seen the model reproduces the main features of the scattering pattern seen in full simulations - very well.

B. Models for the Sine Godon kinks[4]

In the sine-Gordon case we have looked at old results of the scattering of kinks on point inpurities [7] (who have seen similar trapping/transmission/reflection pattern) and their recent explanation by Goodman and collaborators [10]. In that work Goodman et al explained the observed results by invoking an interaction of the kink with the oscillation of the vacuum (around the impurity). Thus their model involved two degrees of freedom - the position of the kink x_0 and the amplitude of the vacuum oscillation (at the impurity point) a. The model of Goodman et al reproduced all the features of the results of the original simulations reported in [7] very well and so our two models discussed in [4] are the adaption of the ideas of Goodman et al to our case. In both models we introduce degrees of freedom describing the standing waves in the hole (in one model the waves are restricted to lie at the edges of the hole and in the other they describes the waves in the hole). In both models we have chosen our waves somewhat arbitrarily. The idea was not to reproduce the pattern in any detail but just to see whether the mechanism of Goodman works in this case too.

In fact the models work surprisingly well. Both reproduced the pattern very well indeed, although the critical value of velocity was a little too high. Given that these models involved only 3 or 4 degrees of freedom the results were very positive and encouraging further work to understand better which modes are important and which are less so.

V. PRELIMINARY RESULTS ON BREATHERS

The Sine-Gordon model, in addition to kink, also possesses breathers as its solutions. These breathers are given by

$$\phi(x,t) \;=\; 2\,arctan\left(\frac{\sin(\omega t)}{\omega\,\cosh(\sqrt{1-\omega^2}\,x)}\right).$$

Their energy is $16\sqrt{1-\omega^2}$ and they are often thought of as bound states of a kink and antikink with the binding increasing as $\omega \rightarrow 1$. Hence it is interesting to see what can happen as a breather is sent towards a well; can it scatter by changing its ω and can it split leaving a kink (or an antikink) in the well and allowing its partner to scatter forwards or backwards.

We have performed several numerical ezperiments of such systems and we plan to present our results soon [11]. Our preliminary results show that the breathers can get trapped, pass with, in general a different ω (note that increasing ω releases some energy), or split with either a kink or an antikink being ejected from the hole.

In fig 4 we present a couple of pictures showing a breather just before a scattering on a hole, and some time later, in the case when the interaction with the hole led to breather's splitting.

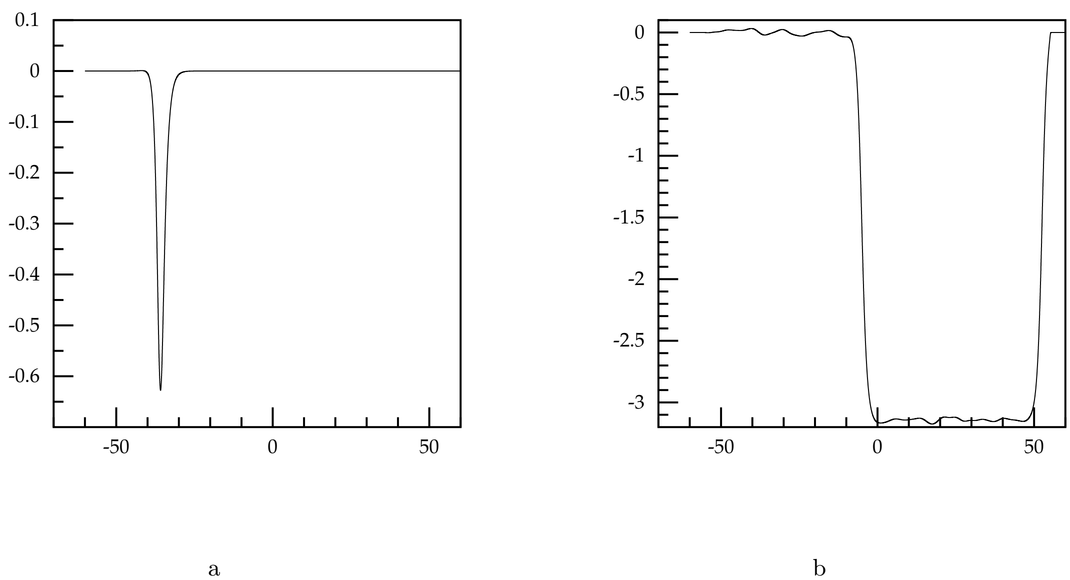

FIG. 4: Trajectory of a breather sent with velocity $v = 0.06587375$ towards a well of width $L = 10$ and depth 0.8 (*ie* $\lambda = 0.8$ a) $t = 60$, b) $t = 800$.

VI. FURTHER COMMENTS AND CONCLUSIONS.

We started by giving a very brief review of topological solitons and of their dynamics. Then we reviewed results of our studies of the scattering of topological solitons on a potential, of both a barrier and a hole-type. We finished by reporting some preliminary results for breathers.

When the soliton was sent towards the barrier its behaviour resembles that of a particle. Thus at low energies the soliton was reflected by the barrier and at higher energy it was transmitted. The scattering process was very elastic. During the scattering the kinetic energy of the soliton was gradually converted into the energy needed to 'climb the barrier'. If the soliton had enough energy to get to the 'top' of the barrier then it was transmitted, otherwise it slid back regaining its kinetic energy.

Note that the soliton size is related to the parameters of the model and so depends on μ. Hence, during the climb of the barrier, the soliton altered its size (it decreased a little) - to fit the local value of μ; when it got through, or slipped back, its size returned to it original value. This is what one would expect in an elastic scattering and this is what we saw in the numerical simulations. In fact, the soliton size oscillated a little, around its 'correct' value and the amplitude of these oscillations has not changed much during the scattering process and the final oscillations resembled the original ones.

In the hole case, the situation was very different. This time, the soliton gained an extra energy as it entered the hole. Some of this energy was converted into kinetic energy of the soliton, some was radiated away. So when the soliton tried to 'get out' of the hole it had less kinetic energy than at its entry and, when this energy was too low it remained trapped in the hole. During the scattering process, like in the case of a barrier, the soliton size changed and its oscillations increased significantly. Afterwards they stayed like this - with much higher amplitude of oscillations than before. Hence the increase in oscillations is related to the inelasticity of the process and the emitted radiation.

Our models, both in one dimensions and in two dimensions, reproduced these results very well.

We also looked at the scattering of breathers on potential holes. As breathers can be thought of as bound states of kinks and antikinks they could split leaving trapping a kink or antikink in the hole and allowing its partner to escape either forwards or backwards. On top of that the energy of the breather depends on the frequency of their oscillation ('breathing') so this could change as well. And, as breathers carry zero topological charge they can also be created in the process (although this costs some energy so such things do not happen too often). All such phenomena have been seen in our simulations and we hope to report on this in the near future.

Finally, all our results generalise to other models; such as *ie* the Landau-Lifschitz model with a position dependent potential or external magnetic field, other models in (1+1) dimensions, such as a $\lambda\phi^4$ model or even models describing ferro- and anti-ferro-magnets[12].

Clearly a lot of work has still to be done in this area.

Acknowledgements

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