Magnetic Braids

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with

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(University of Dundee)

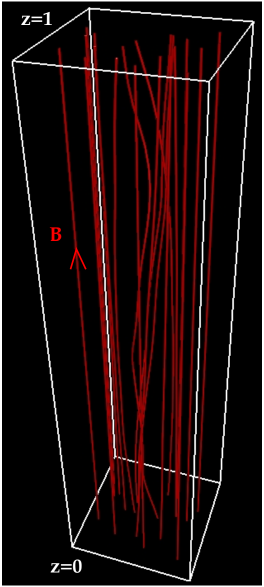
28th October 2011

Numerical Analysis Seminar, Durham

What is a magnetic braid?

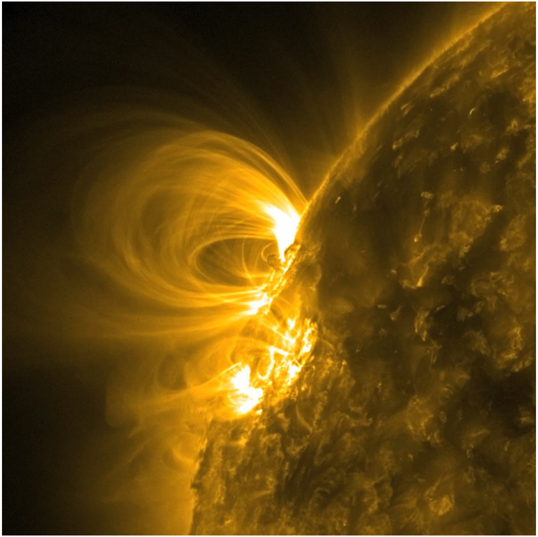
- A magnetic braid is a magnetic field $\mathbf{B}(x,y,z)$ in the space $0 < z < 1$ that satisfies $B_z > 0$.

$$\nabla \cdot \mathbf{B} = 0.$$



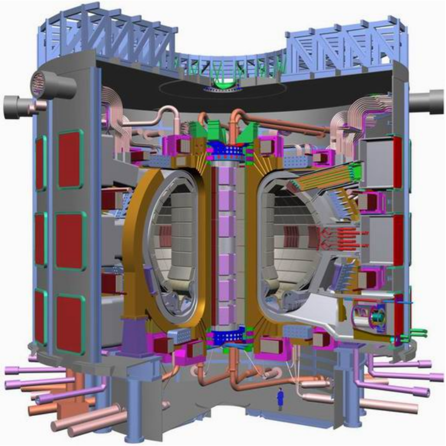
Examples

1. Magnetic loops in the solar corona.

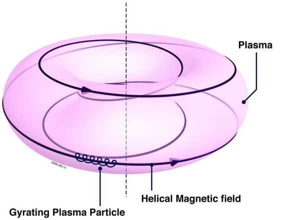


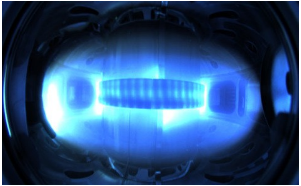
NASA Solar Dynamics Observatory (23 Feb 11).

2. Thermonuclear confinement devices.



ITER (Internat'l Thermonuclear Experimental Reactor).





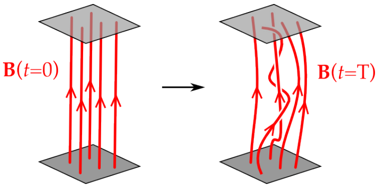
Inside the KSTAR tokamak.

- Correspond to periodic magnetic braids.

Topological equivalence

- Two magnetic braids are topologically equivalent if they are related by an ideal deformation v vanishing on z = 0 and z = 1:

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0.$$



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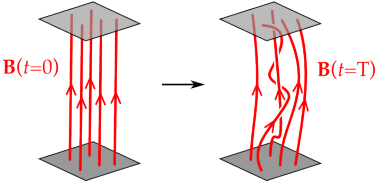
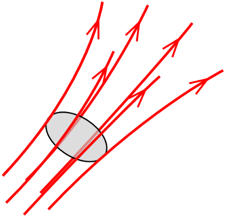




Theorem (Alfvén, 1942)

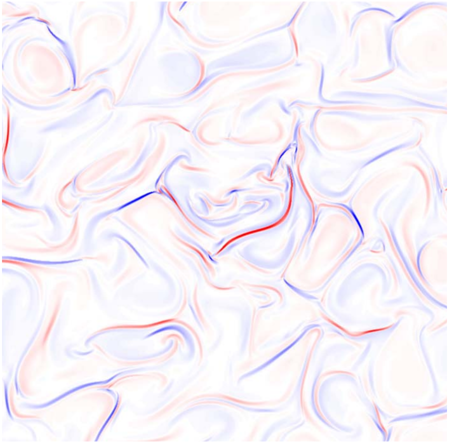
In an ideal evolution the magnetic flux through any co-moving surface is conserved.

$\implies$ conservation of field lines

Hannes Alfvén receiving the Nobel Prize in Physics, 1970.

Physical importance

- Many plasmas are very highly-conducting.
- Changes in topology (magnetic reconnection) occur only in small regions of high ∇B.



Current sheets ($j = \nabla \times B$) from Servidio et al. (2010 Phys. Plasmas.).

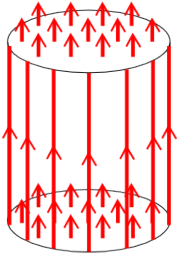
Our problem

1. How do we tell if two magnetic braids are topologically equivalent?
   (necessary and sufficient conditions)

2. How do we quantify differences in their topology?

Assumptions

- To simplify the discussion, we consider a cylinder $0 < r < R$, $0 < z < 1$ with simple boundary conditions:



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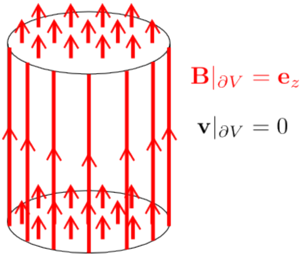

- We parametrise the field lines by z so that

$$\frac{d\mathbf{f}_z(r_0,\phi_0)}{dz} = \frac{\mathbf{B}\big(\mathbf{f}_z(r_0,\phi_0)\big)}{B_z\big(\mathbf{f}_z(r_0,\phi_0)\big)}, \qquad \mathbf{f}_0(r_0,\phi_0) = (r_0,\phi_0).$$

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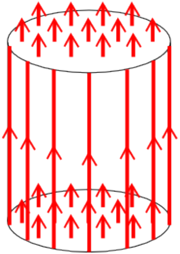

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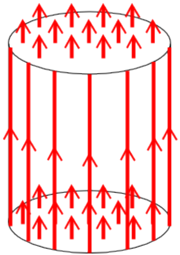

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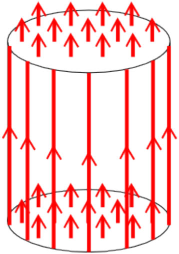

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- So f_1 is the field line mapping from $z = 0$ to $z = 1$.
- With our assumptions, f_1 is a necessary and sufficient condition for topological equivalence. Can we do better?

Magnetic helicity

The magnetic helicity is

$$H = \int_V A \cdot B \, d^3x, \qquad \text{where} \qquad B = \nabla \times A.$$

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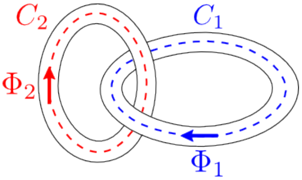
- Measures the linking of magnetic flux:

e.g. two closed thin untwisted tubes,

$$H = \Phi_1 \oint_{C_1} A \cdot dl + \Phi_2 \oint_{C_2} A \cdot dl = \pm 2n \Phi_1 \Phi_2$$

where n is the linking number.



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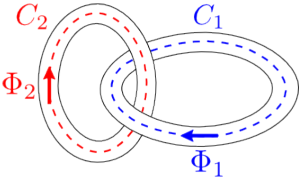

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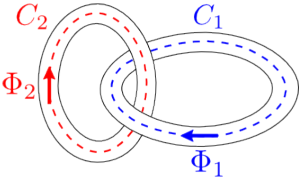

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  - If $B_n|_{\partial V} = v_n|_{\partial V} = 0$, then H would be an ideal invariant $\Longrightarrow$ necessary condition for same topology

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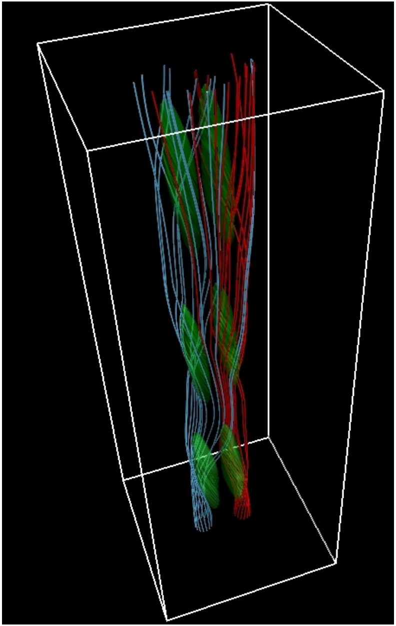
In the gauge $A|_{\partial V} = \frac{r}{2} e_\phi$, then H is a necessary condition for topological equivalence.

Relaxation of a coronal loop



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  - Wilmot-Smith, Hornig & Pontin (2010, Astron. Astrophys.);
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- $H = 0$ throughout despite changing topology.

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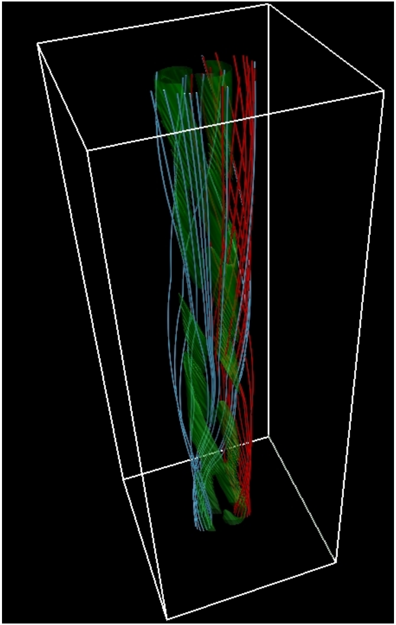

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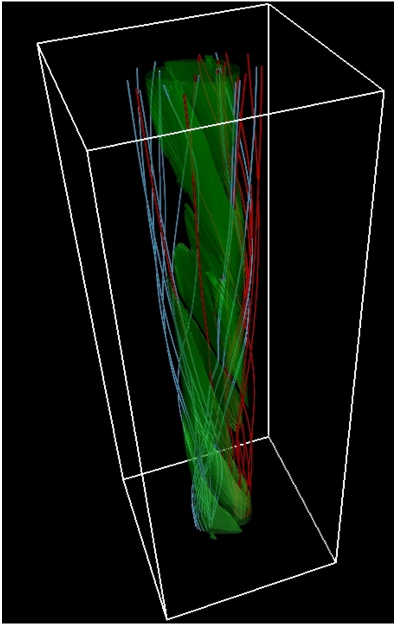

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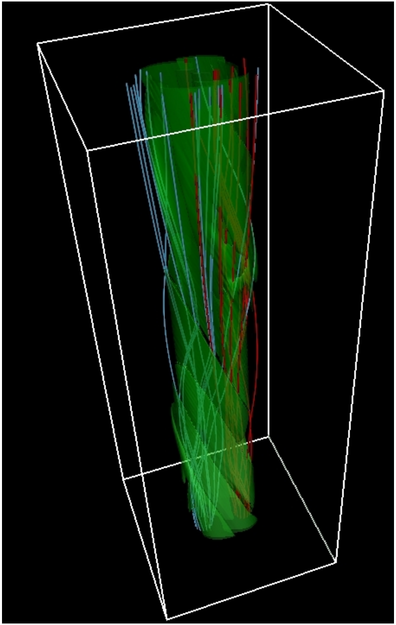

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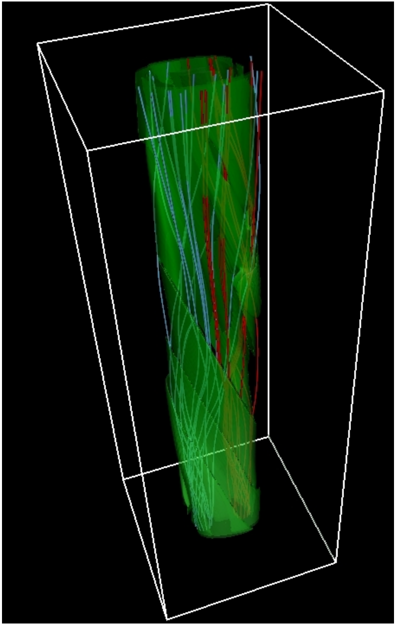

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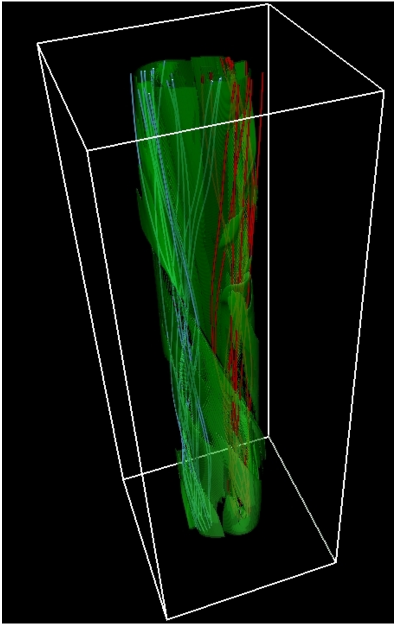

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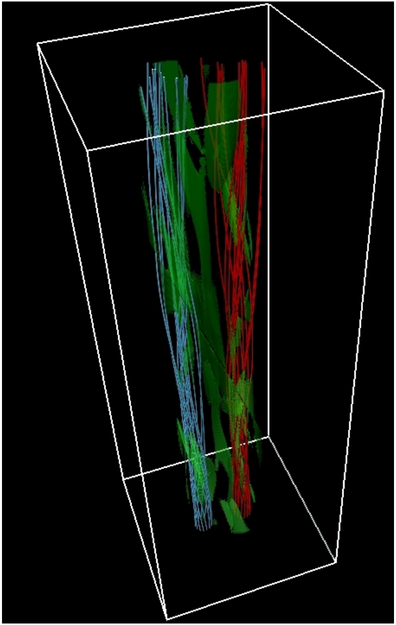

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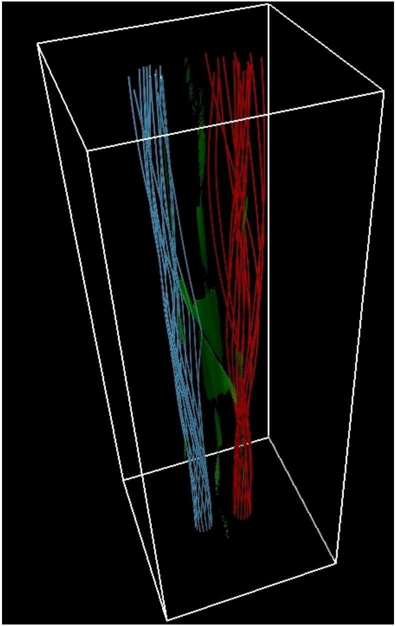

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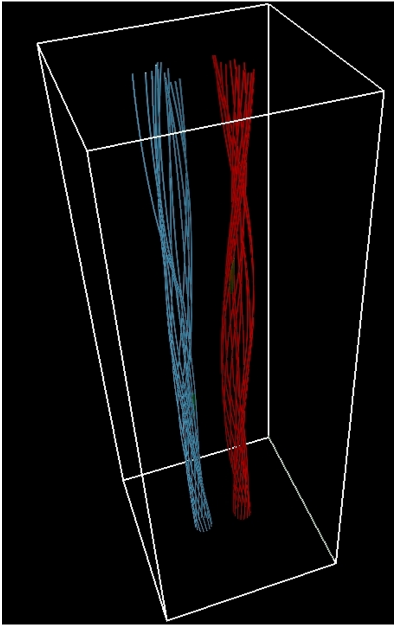

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Alfvén $\Longrightarrow$ a function measuring fluxes through comoving loops will be an ideal invariant.

The topological flux function $\mathscr{A} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\mathscr{A}(r_0, \phi_0) = \int_{z=0}^{z=1} \mathbf{A} \cdot d\mathbf{l},$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ and the integral is along a magnetic field line.

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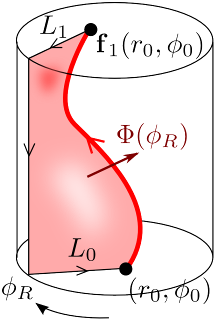

- Poloidal flux

$$\Phi(\phi_R) = \oint \mathbf{A}\cdot d\mathbf{l}$$
$$= \mathscr{A}(r_0, \phi_0) + \int_{L_0} \mathbf{A}\cdot d\mathbf{l} + \int_{L_1} \mathbf{A}\cdot d\mathbf{l}$$
$$- \int_0^1 A_z(R, \phi_R)\, dz$$

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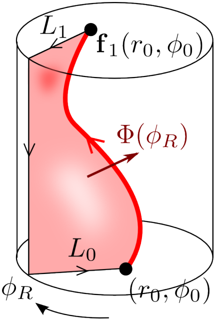


- Poloidal flux

$$\begin{aligned}\Phi(\phi_R) &= \oint \mathbf{A}\cdot d\mathbf{l}\\ &= \mathscr{A}(r_0,\phi_0) + \int_{L_0}\mathbf{A}\cdot d\mathbf{l} + \int_{L_1}\mathbf{A}\cdot d\mathbf{l}\\ &\quad - \int_0^1 A_z(R,\phi_R)\,dz\end{aligned}$$

- $\mathscr{A}(r_0,\phi_0)$ is the mean $\Phi(\phi_R)$ over all angles ϕ_R.

Ideal invariance

- $\mathscr{A}(r_0, \phi_0)$ is an ideal invariant for all (r_0, ϕ_0):

$$
\begin{aligned}
\frac{d\mathscr{A}}{dt} &= \frac{d}{dt}\int_0^1 A \cdot dl \\
&= \int_0^1 \left(\frac{\partial A}{\partial t} - v \times \nabla \times A + \nabla(v \cdot A)\right) \cdot dl \\
&= \int_0^1 \nabla\big(\Phi + v \cdot A\big) \cdot dl \\
&= \big(\Phi + v \cdot A\big)\Big|_{(r_0,\phi_0)}^{f_1(r_0,\phi_0)} \\
&= 0
\end{aligned}
$$

- Uses gauge restriction.

Relation to helicity

- Change variables to (r_0, ϕ_0, z) defined by $(r, \phi, z) = f_z(r_0, \phi_0)$ with Jacobian

$$\det(J) = \frac{r_0 B_z(r_0, \phi_0, 0)}{r B_z(r, \phi, z)}.$$

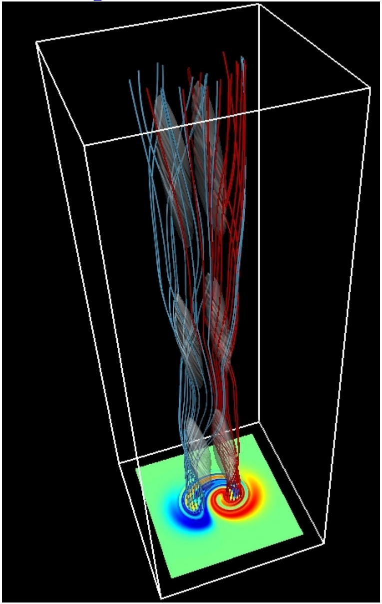
- Then

$$\begin{aligned} H &= \int_V A \cdot B\, r\, dr d\phi dz. \\ &= \int_0^1 \int_{z=0} A\big(f_z(r_0, \phi_0)\big) \cdot B\big(fz(r_0, \phi_0)\big) \frac{B_z(r_0, \phi_0, 0)}{B_z\big(f_z(r_0, \phi_0)\big)}\, r_0 dr_0 d\phi_0 dz \\ &= \int_{z=0} B_z(r_0, \phi_0, 0) \mathscr{A}(r_0, \phi_0)\, r_0 dr_0\, d\phi_0. \end{aligned}$$

- So $\mathscr{A}$ is a field line helicity[1].

[1] Berger (1988, Astron. Astrophys.).

Example



- $\mathscr{A}$ reveals that topology differs from the identity.
- Positive and negative regions cancel so $H = 0$.
- Ideal evolution near boundary $\implies$ persistence of 2 interior critical points.

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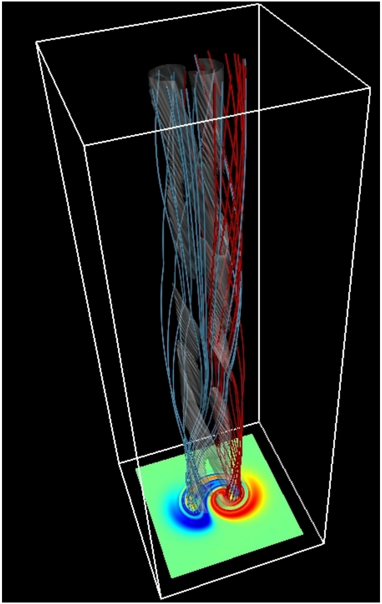

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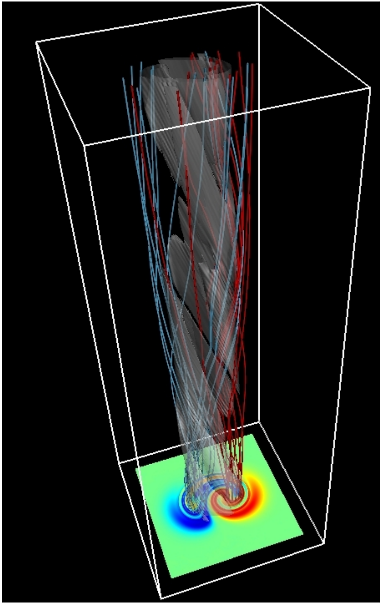

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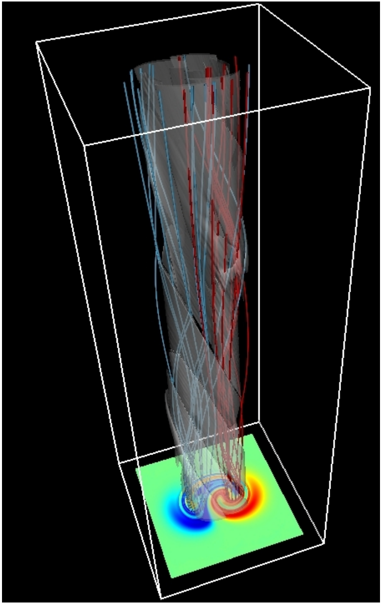

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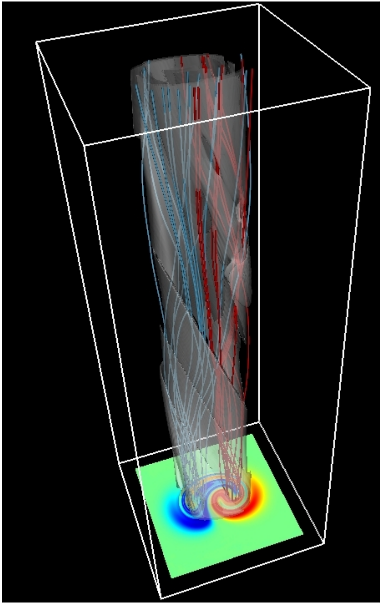

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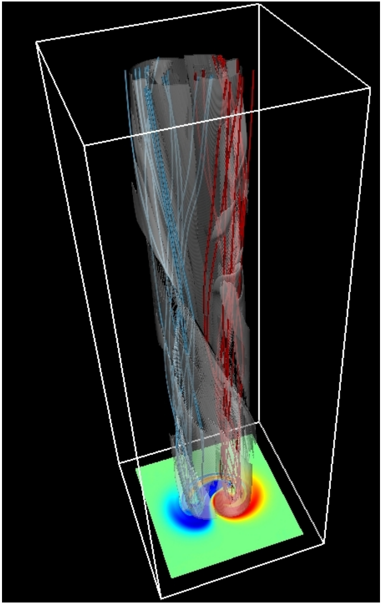

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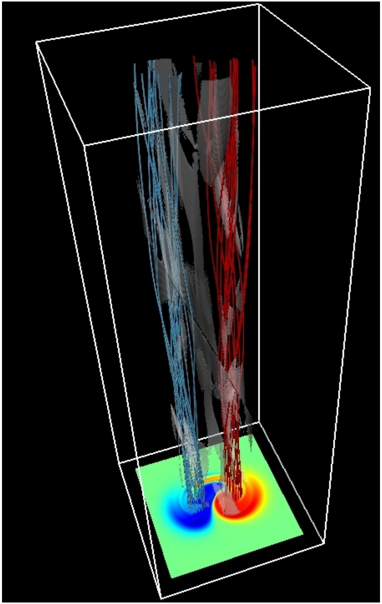

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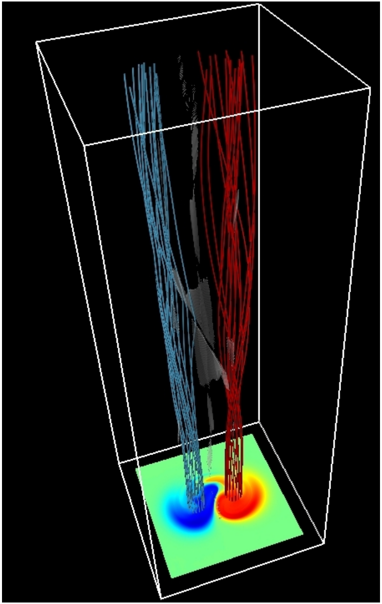

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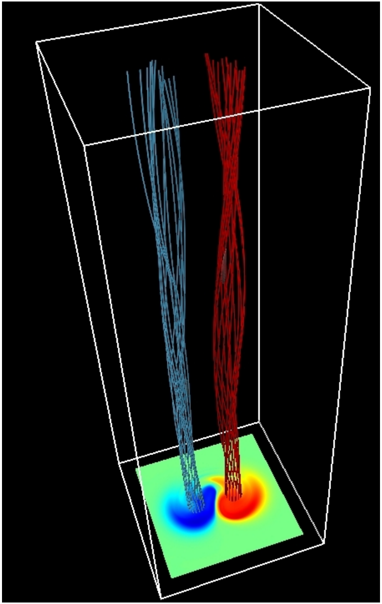

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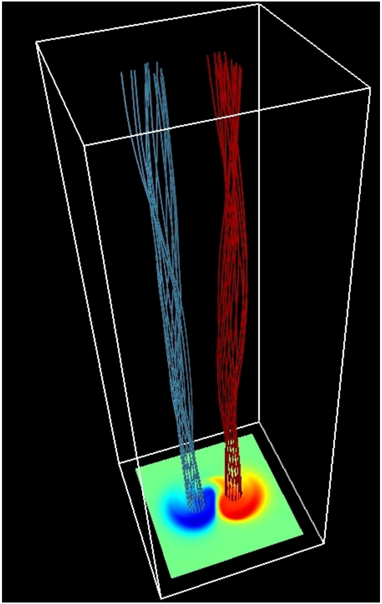

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Another physical interpretation

- The magnetic field lines $\mathbf{f}_l(r_0, \phi_0)$ are given by extremising the action[2]

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  - Euler-Lagrange equations $\implies (\nabla \times \mathbf{A}) \times \frac{d\mathbf{f}_l}{dl} = 0$

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- This is a Hamiltonian system ($z \leftrightarrow$ time), but (r, ϕ) are non-canonical variables.
- Fixing the gauge $A_r = 0$ puts $\mathscr{A}$ in the canonical form

$$\mathscr{A} = \int_0^1 \Big(p\,dq - H(p,q,t)dt \Big)$$

with canonical variables

$$t \leftrightarrow z, \quad p \leftrightarrow rA_\phi, \quad q \leftrightarrow \phi, \quad H \leftrightarrow -A_z.$$

  - f_z preserves phase-space area (magnetic flux).

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Exploiting this analogy

- Tautological/Liouville/canonical 1-form $\alpha = \mathrm{p}\,\mathrm{dq}$.
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Theorem

Take two magnetic braids on the cylinder, with $\mathscr{A}$, $\tilde{\mathscr{A}}$ both in the above gauge. Then

$$\mathscr{A} = \tilde{\mathscr{A}} \iff f_1 = \tilde{f}_1.$$

Proof

Summary

- A magnetic braid is a magnetic field connecting two planes.
- We have introduced a scalar function $\mathscr{A}$ (on a cross-section) that uniquely quantifies the topology under our boundary conditions.
    - More generally, it gives the topology up to a mapping g with $g^*\alpha = \alpha$.
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Future work

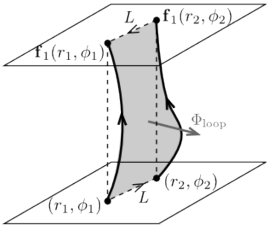
- Using $\mathscr{A}$ to measure reconnection (Yeates & Hornig, 2011).
- What properties of $\mathscr{A}$ are robust under reconnection?
- More general magnetic fields with $B_z \not> 0$?

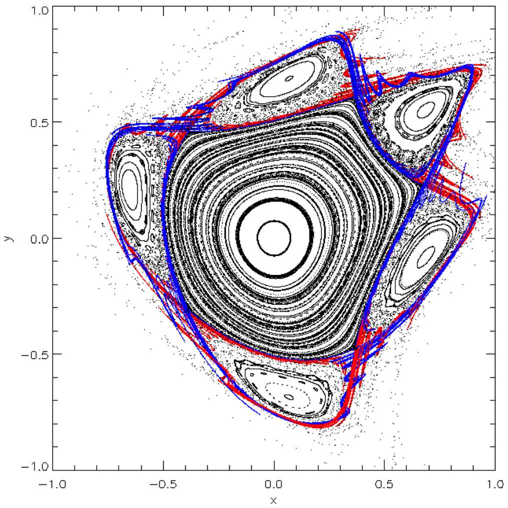
References

- Yeates & Hornig, "A generalised flux function for 3-d magnetic reconnection", Phys. Plasmas (in press).

Measuring reconnection with fixed points

- Yeates & Hornig, "A generalised flux function for 3-d magnetic reconnection", Phys. Plasmas (in press).





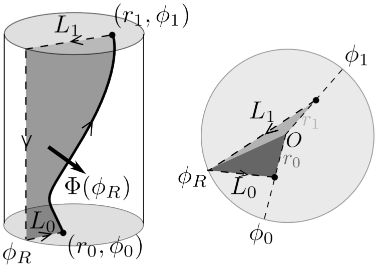
Proof that $\mathscr{A}$ = mean flux

Geometrical argument

Consider the quadrilateral in the $z = 0$ plane with vertices O, (r_0, ϕ_0), (R, ϕ_R), and $(r_1, \phi_1) \equiv f_1(r_0, \phi_0)$.
Since $B_z = 1$ and $A_r = 0$, equating flux through this quadrilateral to its area gives

$$\int_{L_0} A \cdot dl + \int_{L_1} A \cdot dl = \frac{R}{2}\Big[r_1 \sin(\phi_1 - \phi) + r_0 \sin(\phi_0 - \phi)\Big],$$

which vanishes upon averaging ϕ_R from 0 to 2π.



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Outline of proof

- Notation: $f \equiv f_1$, $\tilde{f} \equiv \tilde{f}_1$.

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Boundary conditions on $r = R \implies g = \mathrm{id}$. □

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