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On the limitations of magneto-frictional relaxation

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Motivation

- **Magneto-friction**: ideal induction equation $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

  with artificial "frictional" velocity $\mathbf{u} = \nu^{-1}\mathbf{J} \times \mathbf{B}$

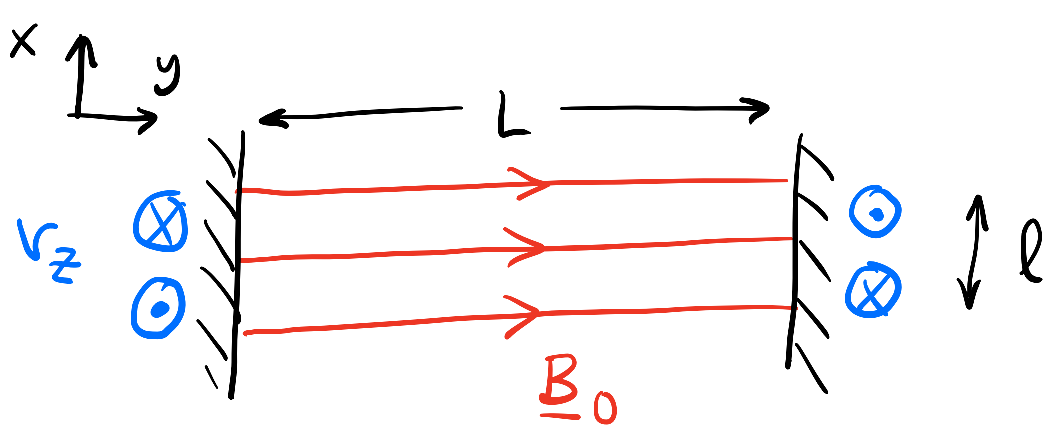
- Leads to monotonic relaxation towards a force-free equilibrium:

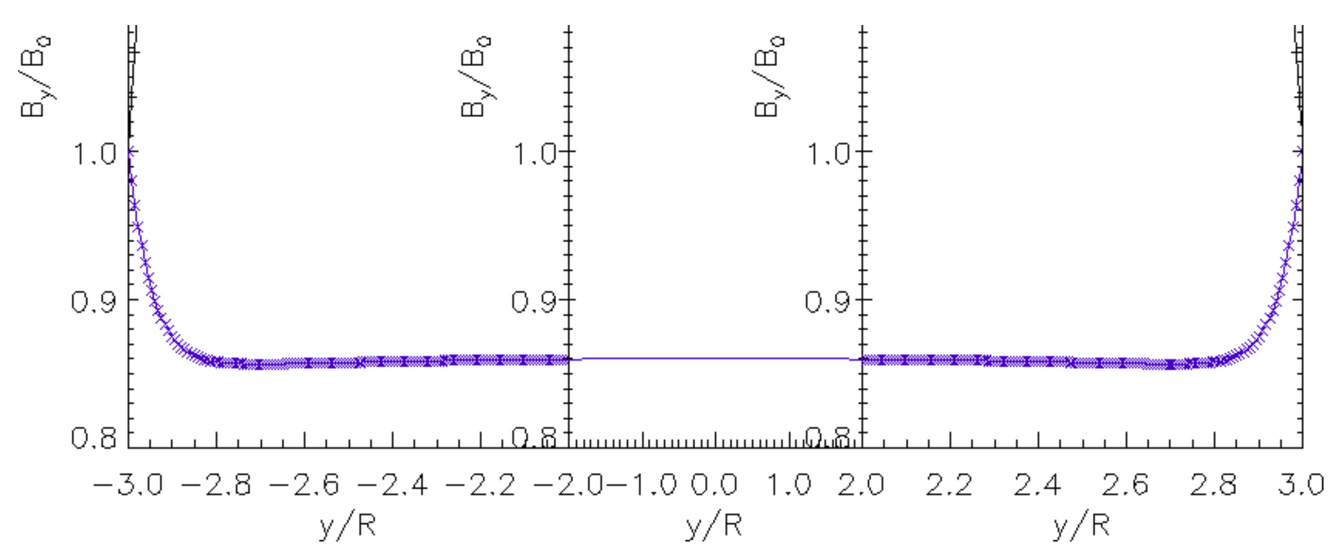
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{B^2}{2\mu_0}\,\mathrm{d}V = -\int_V \nu^{-1}|\mathbf{J}\times\mathbf{B}|^2\,\mathrm{d}V - \oint_{\partial V} \nu^{-1}B^2\mathbf{J}\times\mathbf{B}\cdot\mathrm{d}\mathbf{S}$$

dates at least back to Chodura & Schlüter, *J Comp Phys* [1981]

- Motivation for this study:

  1. Recent comparison of relaxation methods for coronal loop footpoint shearing experiment – Goldstraw et al., *A&A* [2018]
     - MF gives excellent match to full MHD (Lare2D) for low plasma-beta.





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  2. Some objections to the method – Low, *ApJ* [2013]

     (i) Null points cannot move.

     (ii) Discontinuous current sheets will form at null points in finite time.

Aim – test the MF method in a simple case that includes null points.

Test design

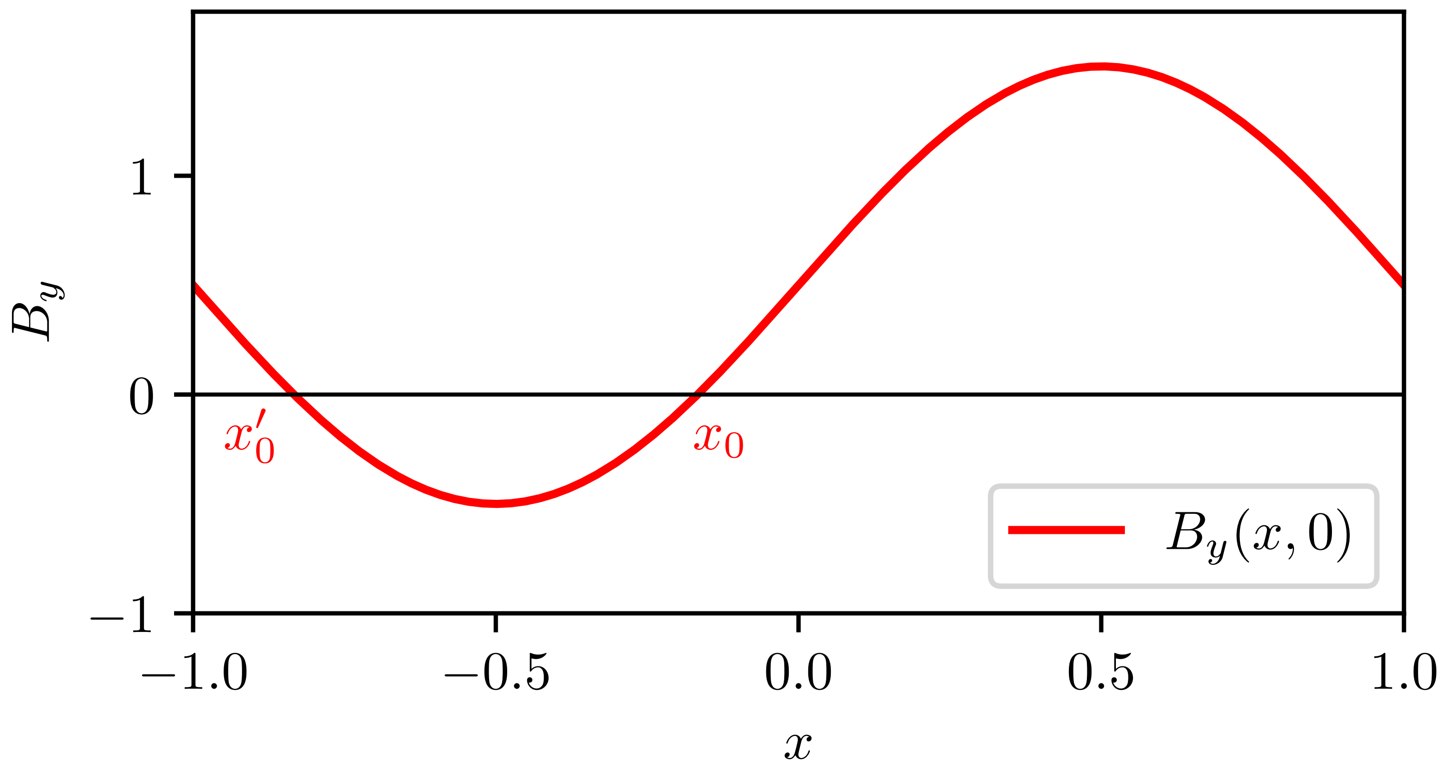
- 1D magnetic field [known "target" solution; sure of controlling numerical diffusion]:
  $\mathbf{B} = B_y(x,t)\mathbf{e}_y$ on periodic domain $-1 \leq x \leq 1$

- Initial condition:

$$B_y(x,0) = \frac{1}{2} + \sin(\pi x)$$

with nulls

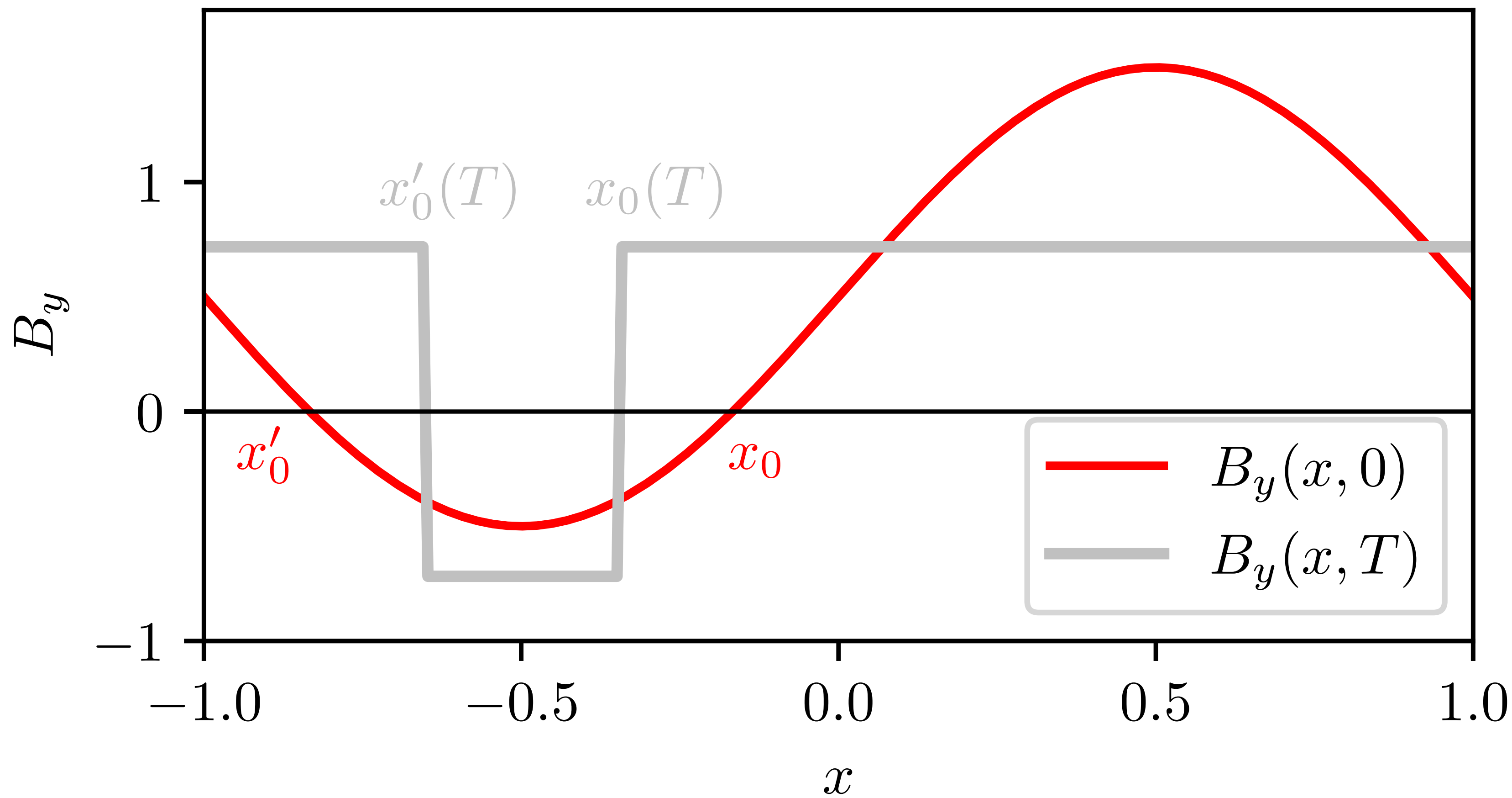
$$x_0 = -\frac{1}{6}, \quad x_0' = -\frac{5}{6}$$



Finding 1 – current sheets *should* form

- For our 1D field, $$\mathbf{J}\times\mathbf{B} = -\frac{\partial}{\partial x}\left(\frac{B_y^2}{2\mu_0}\right)\mathbf{e}_x$$
- So force-free equilibria satisfy $B_y = \pm B_0$ **cf.** Bajer & Moffatt, *ApJ* [2013]
- Ideal relaxation would imply conservation of fluxes between the nulls. Can show that

$$B_0 = \frac{1}{6} + \frac{\sqrt{3}}{\pi} \quad \text{with nulls} \quad x_0(T) = \frac{-1+\Delta}{2}, \quad x_0'(T) = \frac{-1-\Delta}{2}, \quad \Delta = \frac{6\sqrt{3}-2\pi}{6\sqrt{3}+\pi}$$



MF solution in 1D

- In 1D the MF equation reduces to the nonlinear diffusion equation

$$\frac{\partial B_y}{\partial t} = \frac{1}{\mu_0}\frac{\partial}{\partial x}\left(\frac{B_y^2}{\nu}\frac{\partial B_y}{\partial x}\right)$$

(i) linear diffusion $\nu = B_y^2$

(ii) "ambipolar diffusion" $\nu = 1$

(iii) linear diffusion with limiting $\nu = B_y^2 + \varepsilon \mathrm{e}^{-B_y^2/\varepsilon}$

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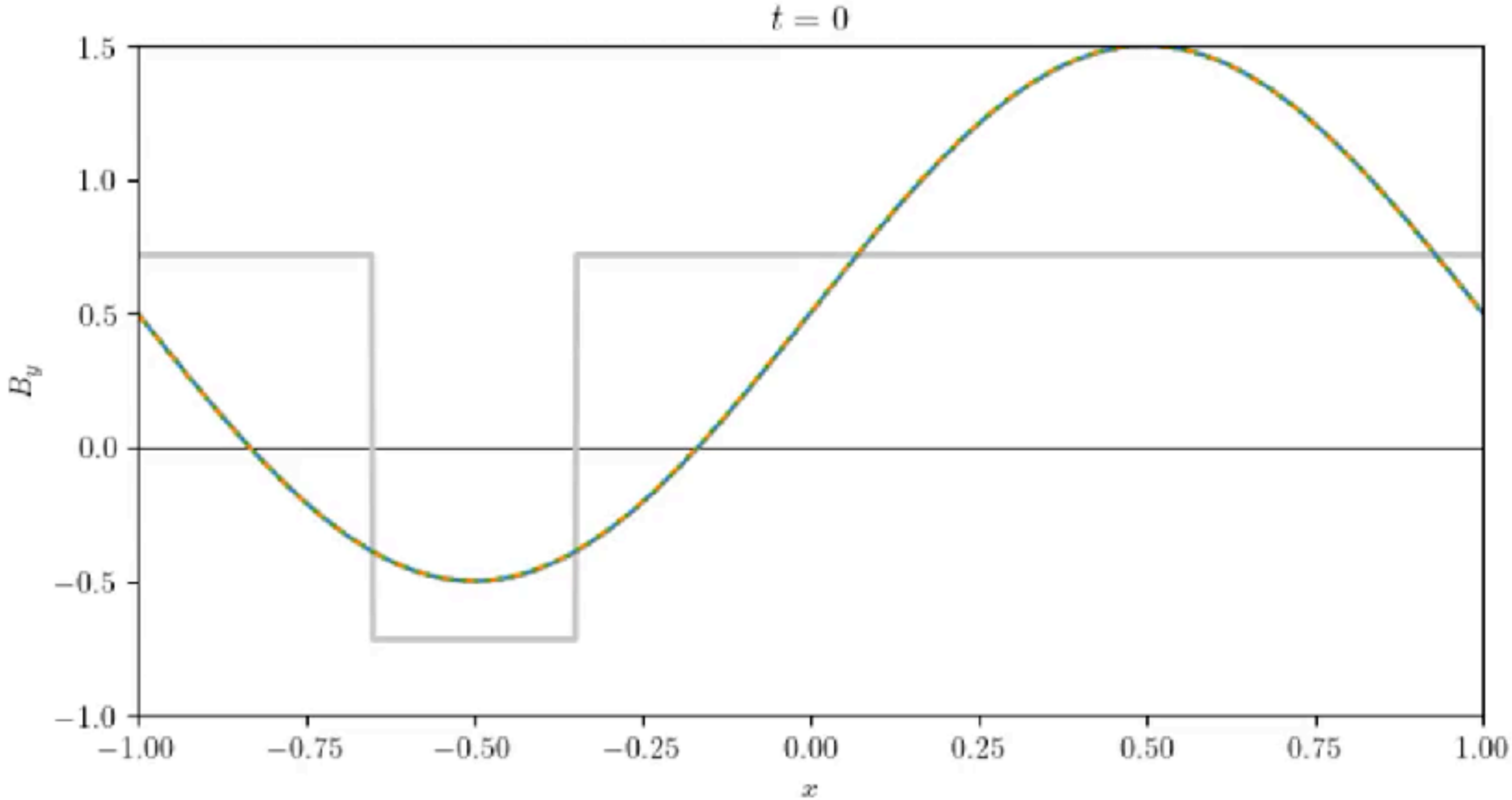


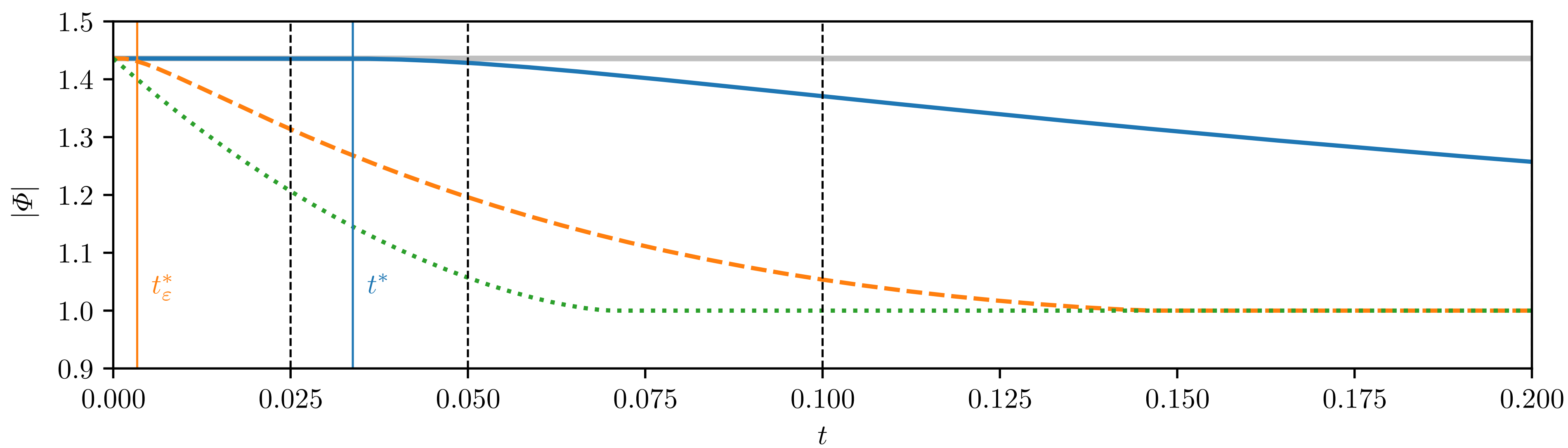
- Numerics: Crank-Nicolson method with Picard iteration for nonlinearity.

Finding 2 – flux is not conserved!

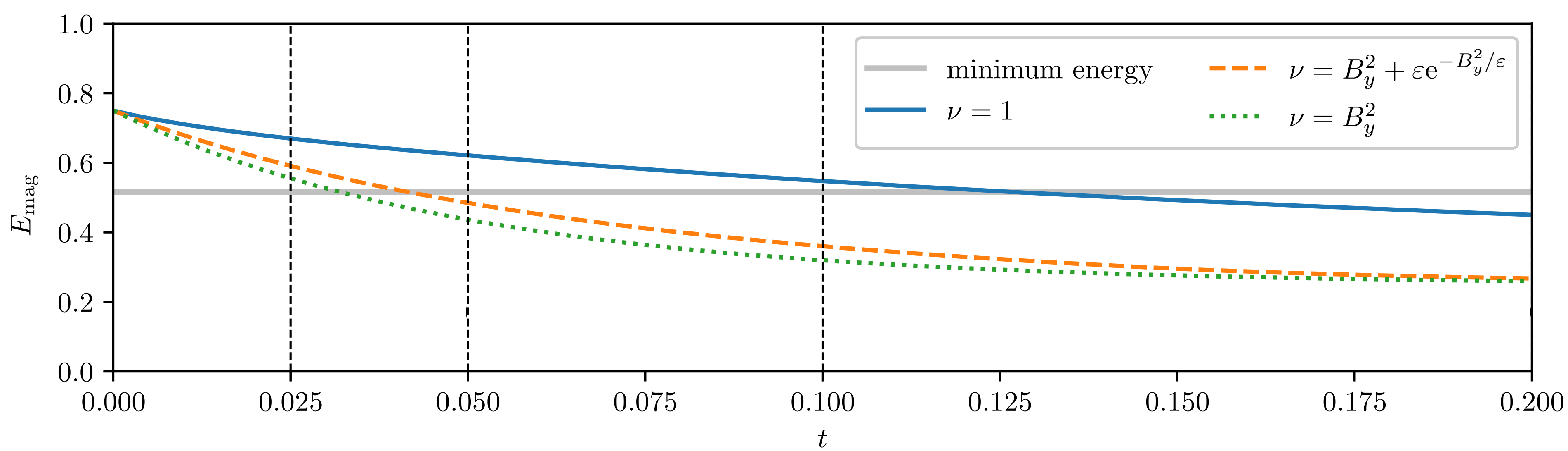
- Linear case: diffusion always gives flux cancellation at nulls.
- Nonlinear cases: flux is conserved *only until the solution becomes discontinuous.*

cf. Hoyos et al, *MNRAS* [2010]

unsigned flux



energy

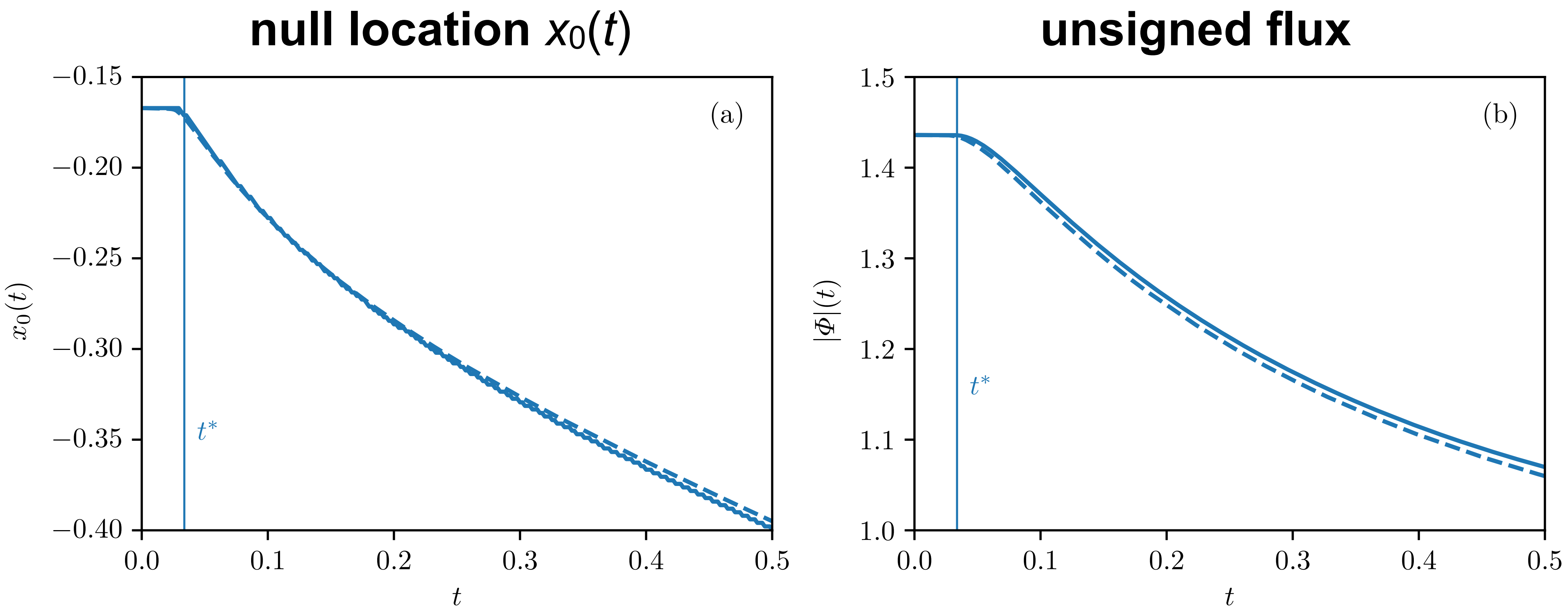


- Breakdown time (**cf**. Low/Aly):

$$t^* = \left[4\left(\frac{\partial B_y}{\partial x}\right)^2_{(x_0,0)}\right]^{-1} \qquad t^* = \varepsilon\left[4\left(\frac{\partial B_y}{\partial x}\right)^2_{(x_0,0)}\right]^{-1}$$

Finding 3 – nulls *can* move

- Linear case: diffusion causes nulls to move in general.
- Nonlinear cases: "nulls" (current sheets) move *once the solution becomes discontinuous.*
  - e.g. ambipolar diffusion case



  - dashed curves show predictions from Rankine-Hugoniot/jump conditions with estimated jumps [see paper]

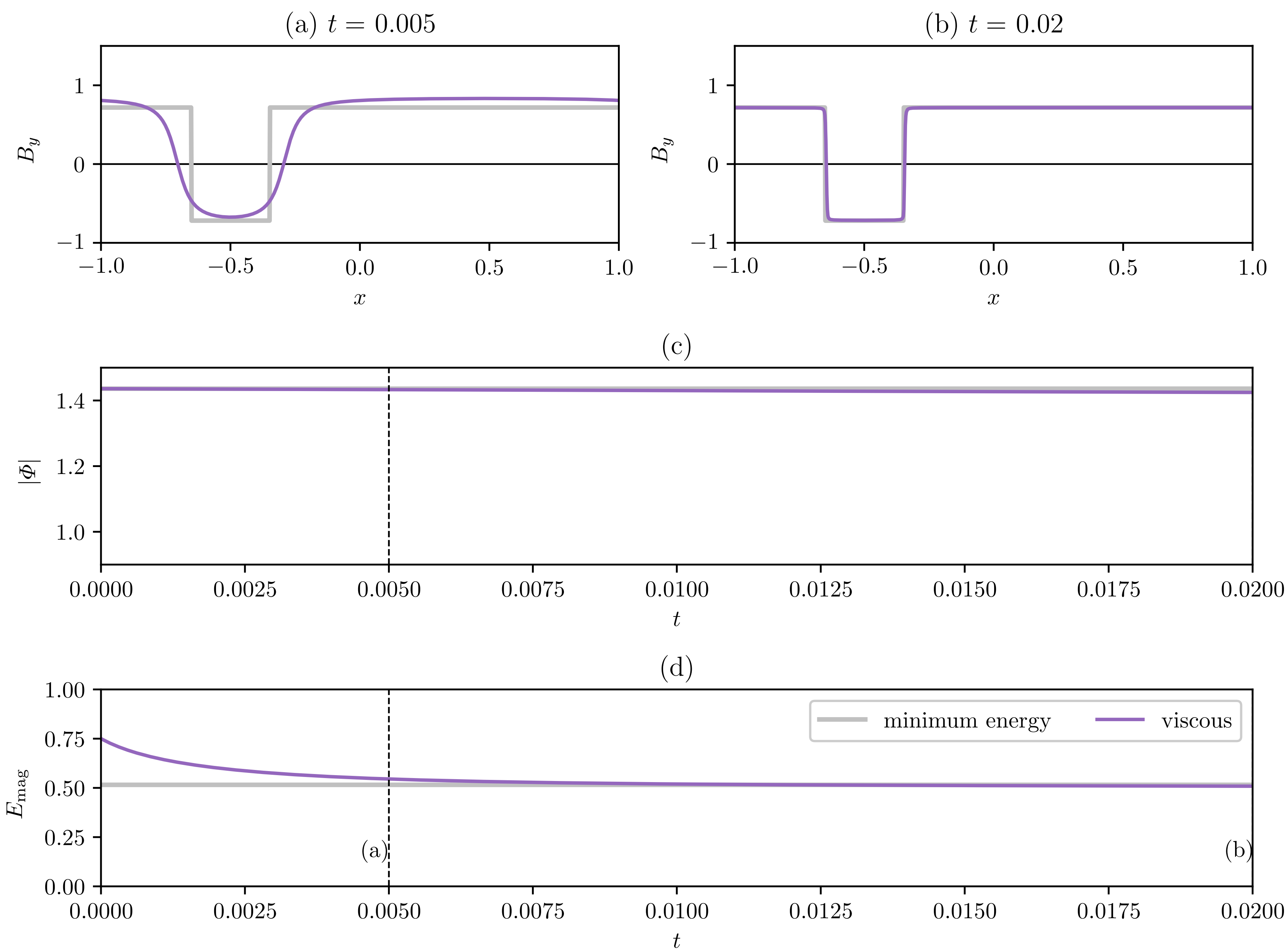
Possible solution: viscous relaxation

- An alternative "viscous relaxation" scheme converges neatly.

$$\frac{\partial B_y}{\partial t} = -\frac{\partial}{\partial x}(u_x B_y)$$

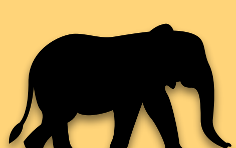
$$\mu \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{B_y^2}{2\mu_0}\right) = 0$$

Bajer & Moffatt, *ApJ* [2013]



- Also ensures monotonic energy decay: $\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{B^2}{2\mu_0}\,\mathrm{d}V = -\mu \int_V \left|\frac{\partial u_x}{\partial x}\right|^2 \mathrm{d}V$

The elephant in the room…



The relaxed state under ideal-MHD would not be force-free.

Ideal-MHD relaxed state

- Now the relaxed state is a total pressure balance

$$\frac{\partial}{\partial x}\left(p + \frac{B_y^2}{2\mu_0}\right) = 0$$

- Gas pressure builds up at the nulls to counter the low magnetic pressure.

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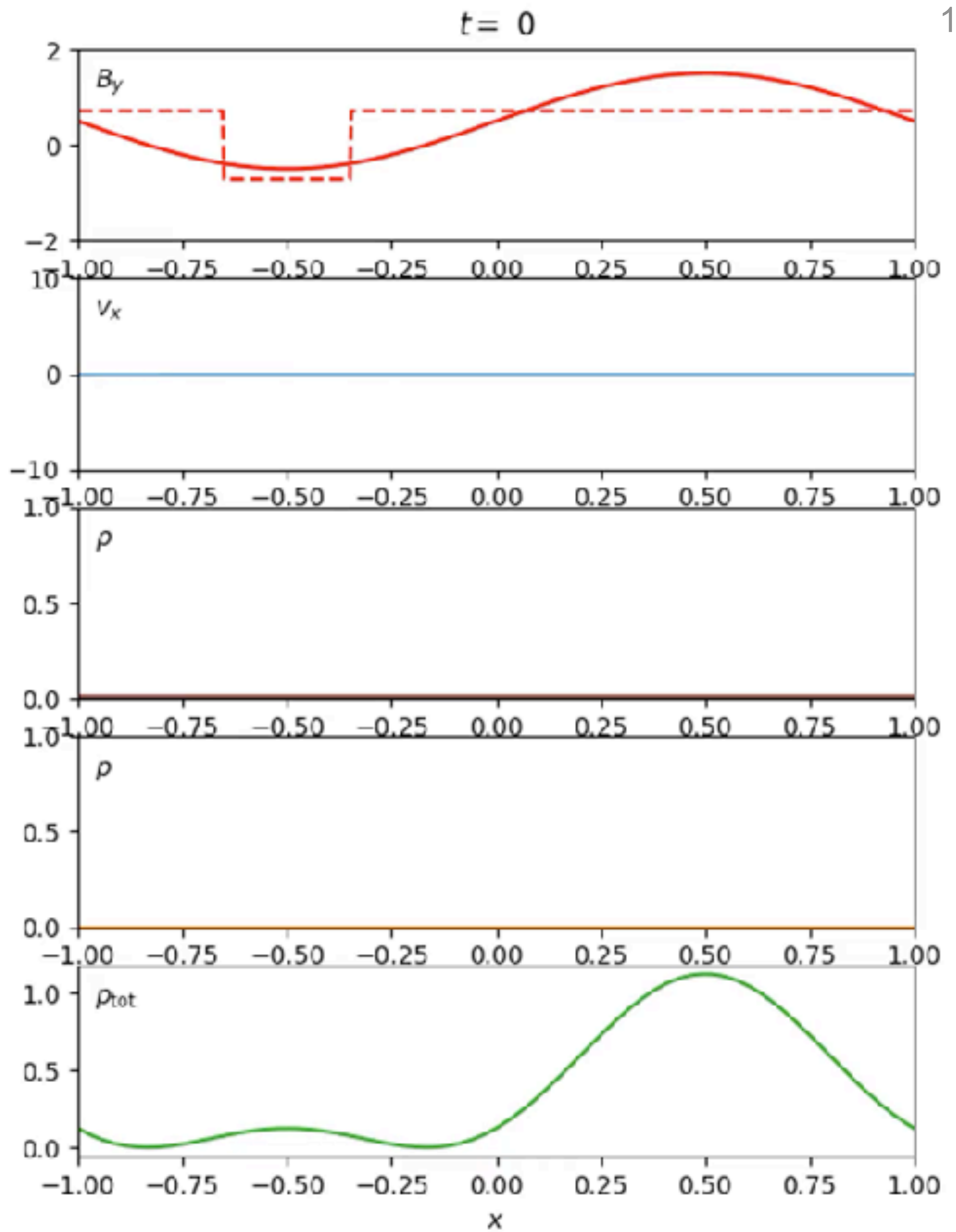
- Solution computed with ATHENA code https://princetonuniversity.github.io/Athena-Cversion/

  [shown with some viscosity but similar conclusion without it]



More details: A.R. Yeates, “On the limitations of magneto-frictional relaxation”, *GAFD*, 2022. https://doi.org/10.1080/03091929.2021.2021197