The Internal Topology of Magnetic Flux Tubes

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with

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Hmmm, when I take the curl of the Navier Stokes Equation, I get the Kelvin Circulation Theorem, which I use to prove Helmholt'z Theorem, ... but then again in a carefully isolated system it will rotate in the other direction in the other Hemisphere!
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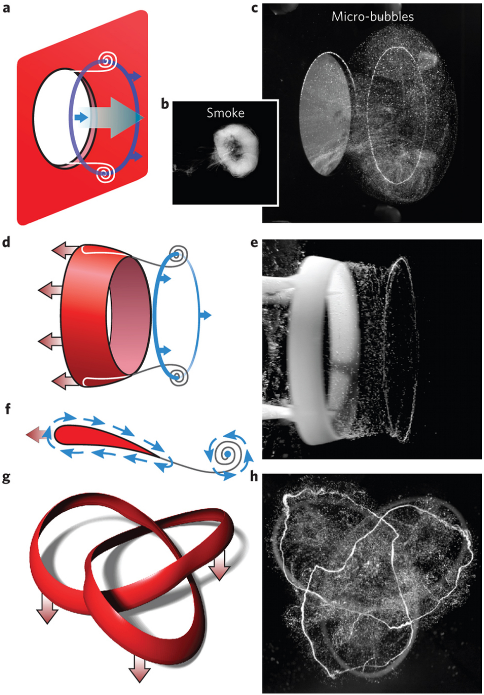

Kleckner & Irvine 2013, *Nature Phys.* **9**, 253.

Vorticity equation (inviscid, barotropic fluid):

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times \left(\mathbf{v} \times \boldsymbol{\omega} \right) \qquad \Longrightarrow \qquad \frac{d}{dt} \int_{S(t)} \boldsymbol{\omega} \cdot \mathbf{n} \, d^2 x = 0$$

(Kelvin's Theorem)

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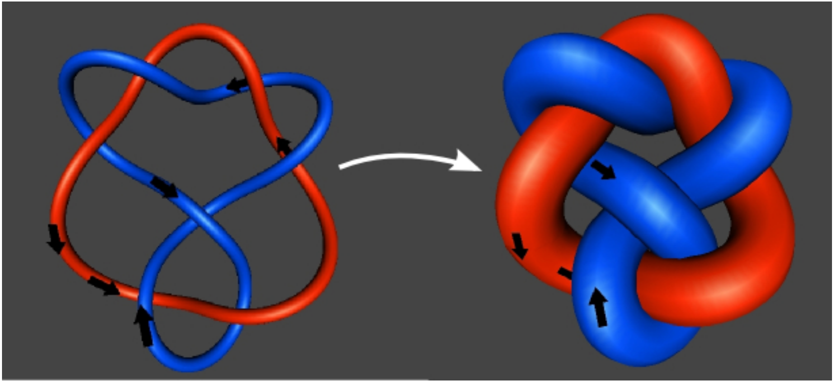
Induction equation (ideal magnetohydrodynamics):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\Big(\mathbf{v}\times\mathbf{B}\Big) \qquad \Longrightarrow \qquad \frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot\mathbf{n}\, d^2x = 0 \quad \text{(Alfvén's Theorem)}$$

Hannes Alfvén receiving the Nobel Prize in Physics, 1970.



after Moffatt 1985, *JFM* **159**, 359.

Magnetic helicity

$$H = \int_V \mathbf{A} \cdot \mathbf{B} \, d^3x, \qquad \text{where} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

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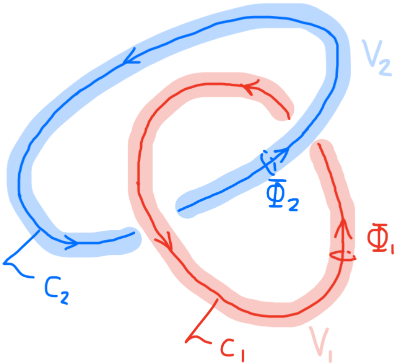
$\Longrightarrow$ H is an ideal invariant in a closed magnetic volume $v_n|_{\partial V} = B_n|_{\partial V} = 0$.

For two thin flux tubes,



$$H = \int_{V_1} \mathbf{A} \cdot \mathbf{B} \, d^3x + \int_{V_2} \mathbf{A} \cdot \mathbf{B} \, d^3x$$

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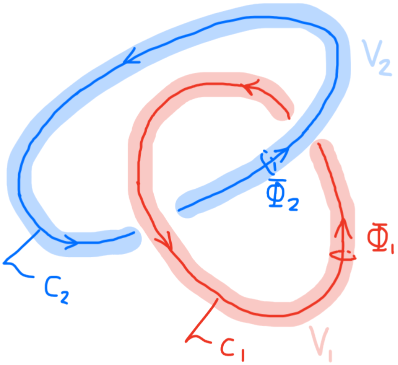

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$$\downarrow \qquad \mathbf{B}\, d^3x \approx \Phi\, d\mathbf{l}$$

$$= \Phi_1 \oint_{C_1} \mathbf{A}\cdot\, d\mathbf{l} + \Phi_2 \oint_{C_2} \mathbf{A}\cdot\, d\mathbf{l}$$

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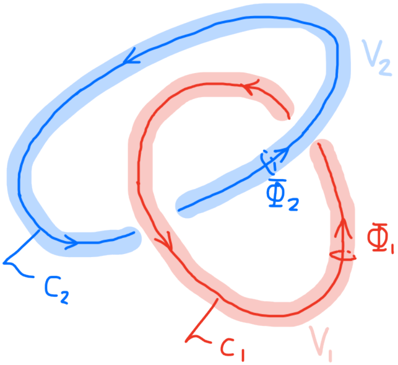

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$$= 2\Phi_1\Phi_2.$$

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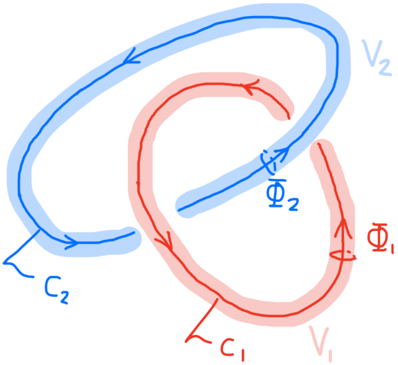

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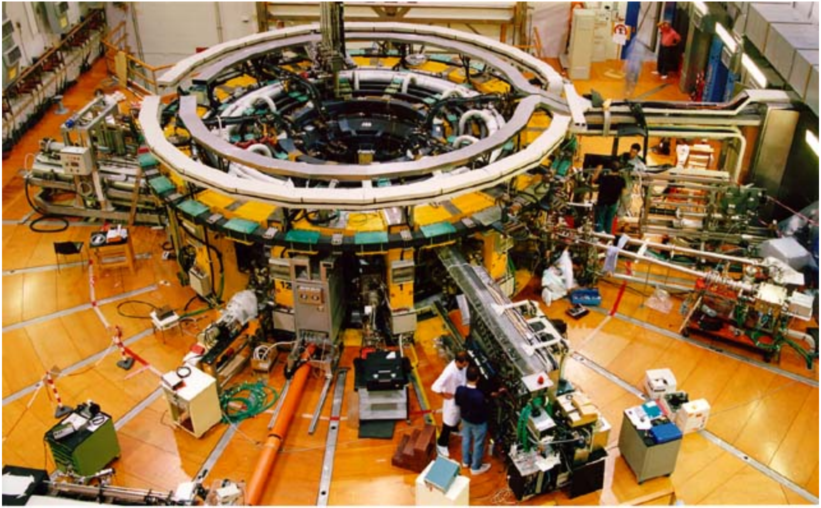
For a collection of discrete flux tubes,

$$H = 2\sum_{i<j} Lk(C_i, C_j)\Phi_i\Phi_j.$$

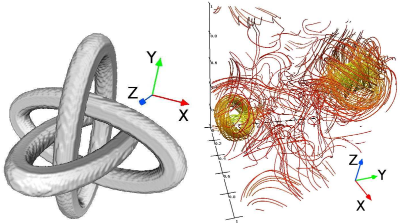
Moffatt 1992, *Proc. R. Soc. Lond. A* **439**, 411.



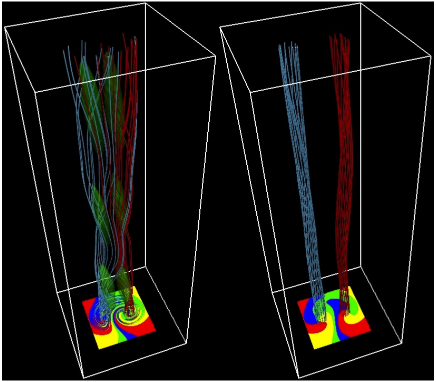
Taylor 1974, *PRL* **33**, 1139 $\rightarrow$ H is the only surviving invariant

Counterexamples?

1. Candelaresi & Brandenburg 2011, *PRE* **84**, 01646.



2. Pontin et al. 2011, *A&A* **525**, A57.

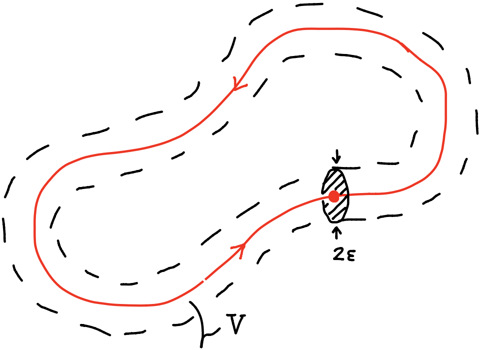


Aims

1. **How do we quantify 3D reconnection?**

2. **What (quasi)-invariants play a role in magnetic relaxation?**

Field line helicities



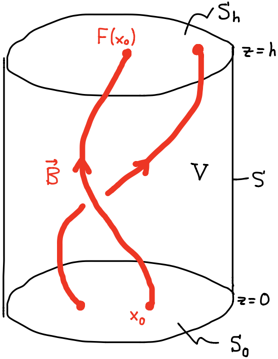
$$\lim_{\epsilon \to 0} \int_{V(\epsilon)} \mathbf{A} \cdot \mathbf{B} \, d^3x.$$

Taylor 1986, *Rev. Mod. Phys.* **58**, 741.

Berger 1988, *A&A* **201**, 355. → energy formula for force-free fields

Our flux tube

$B_z > 0$



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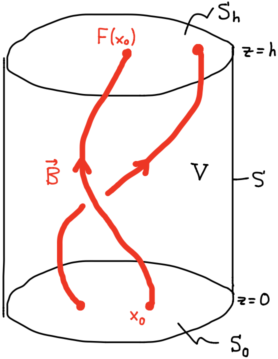

The relative helicity

$$H_{\mathbf{B}'}(\mathbf{B}) = \int_V (\mathbf{A} + \mathbf{A}') \cdot (\mathbf{B} - \mathbf{B}')\, d^3x, \qquad \text{where} \quad B'_n|_{\partial V} = B_n|_{\partial V},$$

is independent of the gauges of $\mathbf{A}$, $\mathbf{A}'$ and ideal invariant for $\mathbf{v}|_{\partial V} = 0$.

Berger & Field 1984, *JFM* **147**, 133.

Finn & Antonsen 1985, *Comm. Plasma Phys. Contr. Fusion* **9**, 111.

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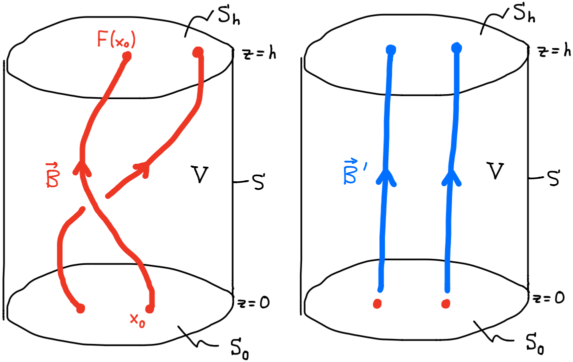

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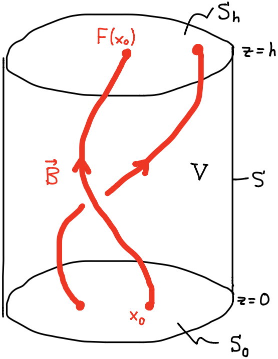
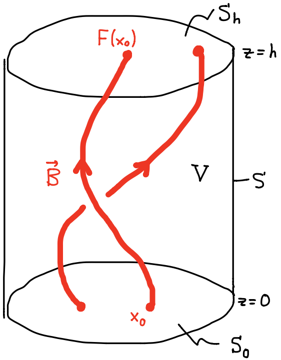
Topological flux function



$$\mathcal{A}(x_0) = \int_{F_z(x_0)} \mathbf{A} \cdot d\mathbf{l}.$$

Topological flux function



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$$\int_{S_0} \mathcal{A}(x_0) B_z(x_0) \, d^2 x_0 = \int_{S_0} \int_0^h \frac{\mathbf{A}(x(z)) \cdot \mathbf{B}(x(z))}{B_z(x(z))} B_z(x_0) \, dz d^2 x_0$$

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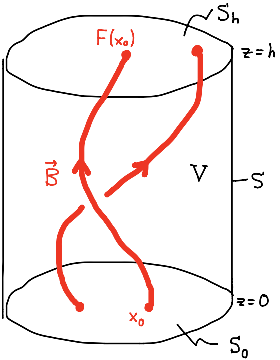

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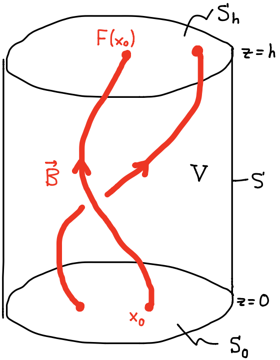

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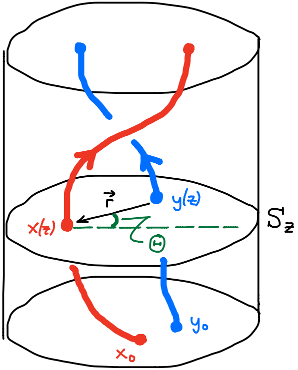
$$\begin{aligned}\int_{S_0} \mathcal{A}(x_0) B_z(x_0)\, d^2x_0 &= \int_{S_0}\int_0^h \frac{\mathbf{A}(x(z))\cdot\mathbf{B}(x(z))}{B_z(x(z))} B_z(x_0)\, dz d^2x_0 \\ &= \int_V \mathbf{A}(x(z))\cdot\mathbf{B}(x(z)) d^3x \\ &= H\end{aligned}$$



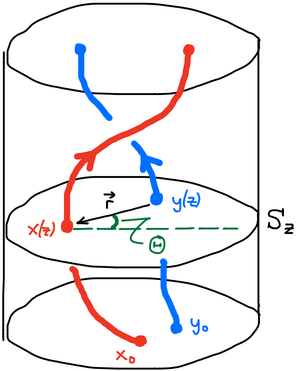
Winding number

$$C(x_0, y_0) = \frac{1}{2\pi} \int_0^h \frac{d}{dz} \Theta(x_0, y_0, z)\, dz$$

where

$$\Theta(x_0, y_0, z) = \arctan\left(\frac{r_2}{r_1}\right),$$

$$\mathbf{r}(x_0, y_0, z) = (x_1 - y_1, x_2 - y_2, 0).$$



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In the Biot-Savart gauge

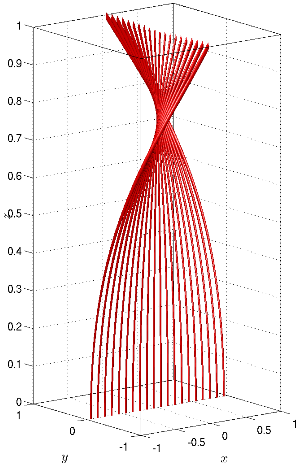
$$\mathbf{A}^{BS}(x) = \frac{1}{2\pi} \int_{S_z} \frac{\mathbf{B}(y_1, y_2, z) \times \mathbf{r}}{|\mathbf{r}|^2}\, d^2 y,$$

we can write

$$\mathcal{A}(x_0) = \int_{S_0} C(x_0, y_0) B_z(y_0)\, d^2 y_0 := \mathcal{A}^{BS}.$$

Simple example

$$\mathbf{B} = B_0\mathbf{e}_z + B_1 rz\mathbf{e}_\phi$$



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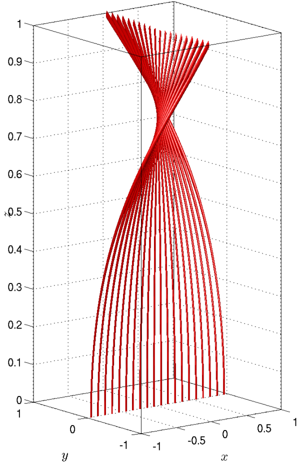
A rare example where we can integrate $\mathbf{A}^{BS}$ explicitly:



$$\mathbf{A}^{BS} = \frac{B_0 r}{2}\mathbf{e}_\phi - \frac{B_1 z}{2}(r^2 - R_0^2)\mathbf{e}_z.$$

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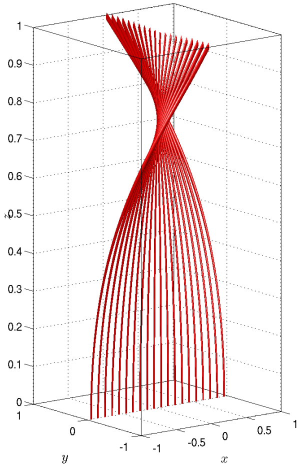

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The field lines are

$$r(z) = r_0, \qquad \phi(z) = \phi_0 + \frac{B_1}{2B_0} z^2,$$

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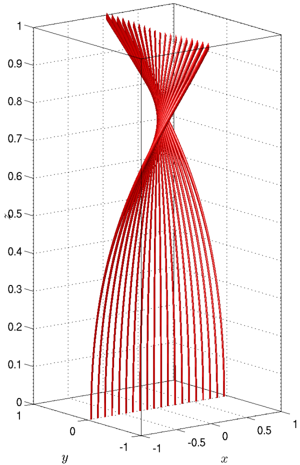

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which have uniform pairwise winding number

$$C(x_0, y_0) = \frac{B_1 h^2}{4\pi B_0}$$

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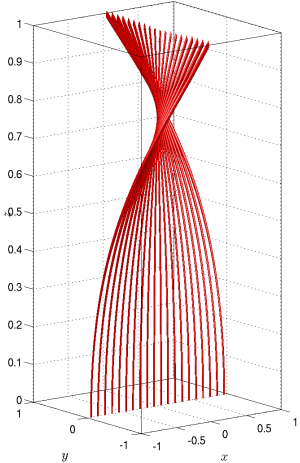

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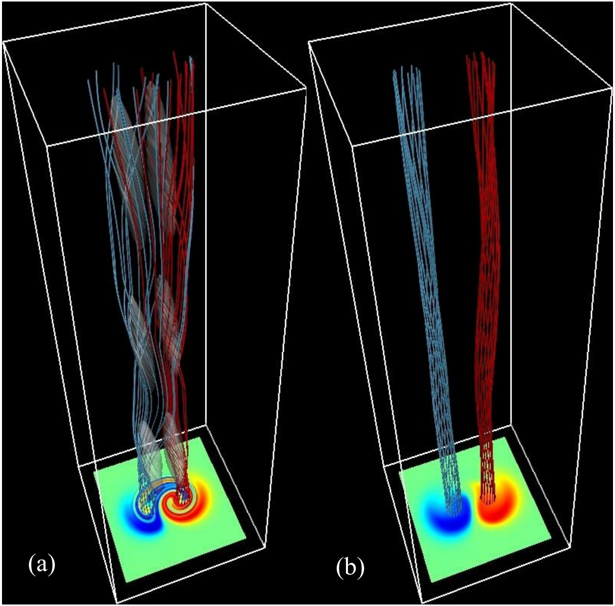
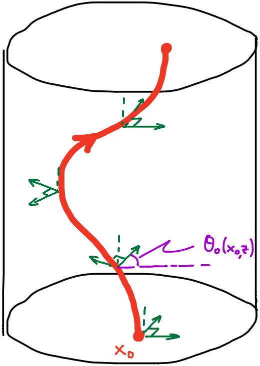
and uniform

$$\mathcal{A}^{BS}(r_0, \phi_0) = \frac{B_1 R_0^2 h^2}{4}.$$


(a)
(b)

Gauge dependence

$$\mathbf{A} \to \mathbf{A} + \nabla\chi \qquad \Longrightarrow \qquad \mathcal{A} \quad \to \quad \mathcal{A} + F^*\chi - \chi$$

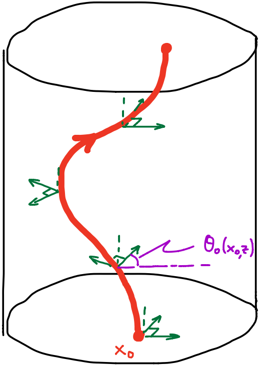


In a rotated frame,

$$r_1' = r_1 \cos\theta_0 - r_2 \sin\theta_0,$$
$$r_2' = r_1 \sin\theta_0 + r_2 \cos\theta_0,$$

we get

$$\Theta' = \arctan\left(\frac{r_2'}{r_1'}\right) = \Theta + \theta_0.$$



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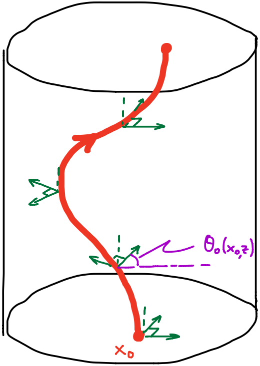
In a general gauge,

$$\mathcal{A}(x_0) = \int_{S_0} C'(x_0, y_0) B_z(y_0)\, d^2 y_0$$

where C' is the winding with respect to some frame field $\theta_0(x_0, z)$.



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Special case: If $\theta_0 = \theta_0(z)$ then
$F^*\chi - \chi = \text{constant} \implies \mathcal{A} = \mathcal{A}^{BS} + \text{constant}$

The Biot-Savart helicity

$$H^{BS}(\mathbf{B}) = \int_V \mathbf{A}^{BS} \cdot \mathbf{B} \, d^3x.$$

For any reference field,

$$H_{\mathbf{B}'}(\mathbf{B}) = H^{BS}(\mathbf{B}) - H^{BS}(\mathbf{B}').$$

cf. Hornig, *A universal magnetic helicity integral*, 2008 (gauge $\nabla_\perp \cdot \mathbf{A}_\perp|_{\partial V} = 0$).

Completeness theorem

(a) Let $\mathbf{B}$, $\mathbf{B}'$ share the same B_n on ∂V, with $\mathbf{n} \times \mathbf{A}'|_{\partial V} = \mathbf{n} \times \mathbf{A}|_{\partial V}$. Then

$$F' = F \text{ with same rotation number} \quad \Longrightarrow \quad \mathcal{A}' = \mathcal{A}.$$

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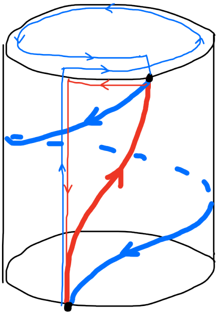
If the difference in rotation numbers $n \in \mathbb{Z}$ is non-zero, then $\mathcal{A}' = \mathcal{A} + n\Phi_0$.



Completeness theorem

(b1) Let $\mathbf{B}$, $\mathbf{B}'$ share the same B_n on ∂V, with $\mathbf{n} \times \mathbf{A}'|_{\partial V} = \mathbf{n} \times \mathbf{A}|_{\partial V}$. Then

$\mathcal{A}' = \mathcal{A}$ for **every** gauge $\iff$ $F' = F$ with same rotation number.

Proof.

We need $(F')^* \chi - F^* \chi = 0$ for an arbitrary gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla \chi$, so must have $F' = F$. □

Completeness theorem

(b2) Let $\mathbf{B}$, $\mathbf{B}'$ share the same B_n on ∂V, with $\mathbf{n} \times \mathbf{A}'|_{\partial V} = \mathbf{n} \times \mathbf{A}|_{\partial V}$, in the specific gauge $A_r|_{S_0,S_h} = 0$. Then

$$\mathcal{A}' = \mathcal{A} \quad \Longleftrightarrow \quad F' = F \text{ with the same rotation number.}$$

Yeates & Hornig 2013, *PoP* **20**, 012102.;Yeates & Hornig 2013, arXiv:1304.8064.

Completeness theorem

(b2) Let $\mathbf{B}$, $\mathbf{B}'$ share the same B_n on ∂V, with $\mathbf{n} \times \mathbf{A}'|_{\partial V} = \mathbf{n} \times \mathbf{A}|_{\partial V}$, in the specific gauge $A_r|_{S_0,S_h} = 0$. Then

$$\mathcal{A}' = \mathcal{A} \quad \Longleftrightarrow \quad F' = F \text{ with the same rotation number.}$$

Proof

Let $G = F' \circ F^{-1}$ and aim to show $G = \mathrm{id}$.

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1. If $A_r = 0$ then $B_z > 0 \implies rA_\phi > 0$ for $r > 0$.

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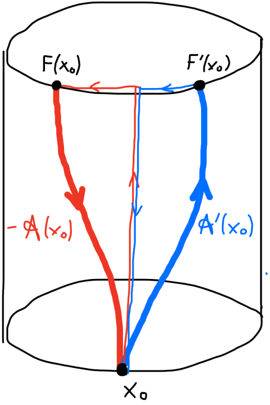
2. Two loops on the boundary imply

$$\int_{F'_\phi}^{F_\phi} rA_\phi \, d\phi = 0,$$

so by step 1, $G|_{\partial S_0} = \text{id}$.



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3. For any curve $\gamma \in S_1$ from x_0 to y_0,

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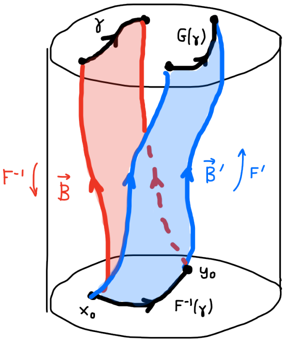

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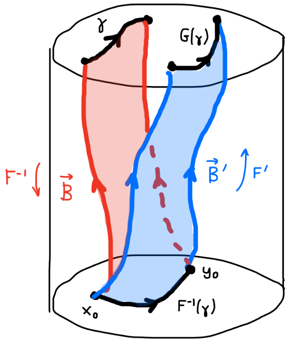
i.e.
$$A_\phi(G(r,\phi)) \frac{\partial G_\phi}{\partial r} = 0, \quad (1)$$
$$G_r A_\phi(G(r,\phi)) \frac{\partial G_\phi}{\partial \phi} - r A_\phi(r,\phi) = 0. \quad (2)$$



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Proof

4. By (1), $G_\phi = g(\phi)$, then the boundary implies $g(\phi) = \phi$.
5. Then (2) gives $G_r A_\phi(G(r,\phi)) = rA_\phi(r,\phi)$. But the Jacobian of the transformation from (r,ϕ) to (rA_ϕ, ϕ) in each plane is non-zero by step 1. Hence $G_r = r$ and $G = \text{id}$. □

Conclusion

We have identified a finer-grained invariant than magnetic helicity and shown that it is complete.

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Ideas we are working on:

- How to measure discrete **reconnection rates** with the flux function? Yeates & Hornig 2011, *PoP* **18**, 102118.
- Which functions of $\mathcal{A}$ are the most robust invariants?
- How to extend to more general magnetic fields?

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The ordinary magnetic helicity is a **meaningful quantity** whichever gauge you choose, but the Biot-Savart gauge is the best choice.

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