# ON GENERAL MAGNETIC RECONNECTION

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## Abstract

**Recent work has shown that a scalar "topological flux function" $\mathcal{A}$ is a complete ideal invariant for a non-null magnetic field between two fixed boundaries (Yeates & Hornig, 2013). In other words, one field can evolve ideally into another with the same boundary condition if and only if they share the same $\mathcal{A}$ function.**

**This poster shows how the usual reconnection rate from GMR theory can be recovered from time changes in $\mathcal{A}$. In addition, $d\mathcal{A}/dt$ encodes spatial information about the reconnection.**

## References

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# 1. Background: General Magnetic Reconnection

**General Magnetic Reconnection was introduced by Schindler *et al.* (1988) as a framework to understand magnetic reconnection in 3D non-null magnetic fields.**

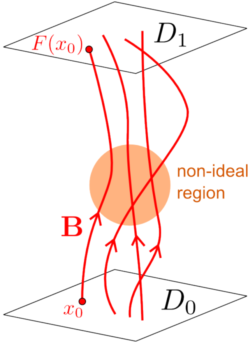
- New field-line connections are made along contours of **integrated parallel electric field**

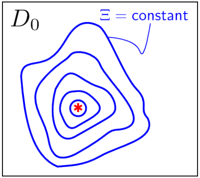
$$\Xi(x_0) = \int_{x_0}^{F(x_0)} \mathbf{E} \cdot \, d\mathbf{l}.$$

  Here $F : D_0 \rightarrow D_1$ is the field-line mapping between boundaries.

- An isolated reconnection site has a maximum $\Xi$ associated to a central **topological field-line** whose connectivity is instantaneously unchanged (Hesse *et al.*, 2005).

- For more complicated reconnection patterns, there will be a set of such "topological field-lines" given by extrema of $\Xi$ (maxima, minima, saddle points).



## 2. Background: Field-line Velocities

**In our non-null field, we can always find a field-line velocity w such that, for some scalar function $\psi$,**

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \nabla\psi.$$

To find **w** (e.g., Roberts, 1967; Hornig, 2007):

1. Calculate $\psi$ by integrating along a field-line:

$$\psi = \int_{x_0}^{F(x_0)} \mathbf{E} \cdot d\mathbf{l}.$$

2. While this fixes the parallel component of $\nabla\psi$, one is free to choose the arbitrary constant on each field-line (equivalently the value of $\psi$ on the initial surface $D_0$).
3. Once $\psi$ is chosen, this determines the perpendicular component

$$\mathbf{w}_\perp = \frac{(\mathbf{E} - \nabla\psi) \times \mathbf{B}}{B^2}.$$

4. Since the perpendicular components of $\nabla\psi$ depend on the initial values chosen, $\mathbf{w}_\perp$ is not unique. But an alternative choice $\psi'$, $\mathbf{w}'_\perp$ must satisfy

$$\mathbf{w}'_\perp \times \mathbf{B} - \nabla\psi' = \mathbf{w}_\perp \times \mathbf{B} - \nabla\psi.$$

5. Irrespective of the choice of $\mathbf{w}_\perp$, the parallel component of **w** is completely arbitrary.

# 3. Topological Flux Function

The **topological flux function** $\mathcal{A} : D_0 \to \mathbb{R}$ is the integral of $\mathbf{A}$ along the field-line:

$$\mathcal{A}(x_0) = \int_{x_0}^{F(x_0)} \mathbf{A} \cdot d\mathbf{l}.$$

**This scalar function encodes the magnetic field structure (Yeates & Hornig, 2013a,b):**

- Given a reference field $\nabla \times \mathbf{A}^{\mathrm{ref}}$, then fixing $\mathbf{n} \times \mathbf{A}|_{\partial V} = \mathbf{n} \times \mathbf{A}^{\mathrm{ref}}|_{\partial V}$ makes $\mathcal{A}$ an **ideal invariant** (for $\mathbf{v}|_{\partial V} = 0$).
- If $H_r$ is the **relative magnetic helicity** then

$$H_r - H^{\mathrm{ref}} = \int_{D_0} \mathcal{A}(x_0) B_z(x_0)\, d^2x_0, \qquad \text{where} \qquad H^{\mathrm{ref}} = \int_V \mathbf{A}^{\mathrm{ref}} \cdot \mathbf{B}^{\mathrm{ref}}\, d^3x.$$

- If $c_{x_0,y_0}$ is the **net winding angle** between two field-lines (Berger, 1988) then

$$\mathcal{A}(x_0) = \int_{D_0} c_{x_0,y_0} B_z(y_0)\, d^2y_0.$$

- Imposing the further "canonical" gauge $A_x^{\mathrm{ref}} = 0$ (or $A_y^{\mathrm{ref}} = 0$) makes $\mathcal{A}$ a **complete invariant**: two magnetic fields with the same $B_z|_{D_{0,1}}$ have the same field-line mapping *if and only if* they have the same $\mathcal{A}$.

# 4. Time Derivative of $\mathcal{A}$

The material derivative of $\mathcal{A}$ under an arbitrary field-line velocity $\mathbf{w}$ is

$$\frac{D\mathcal{A}}{Dt} = -\Xi + \left(\mathbf{w}\cdot\mathbf{A} - \frac{\mathbf{A}\cdot\mathbf{B}}{B_n}w_n - \phi\right)\Bigg|_{x_0}^{F(x_0)}$$

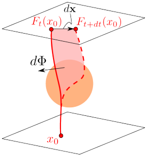
where $\phi$ can be removed by a gauge transformation. To find the time-derivative at a **fixed position**, choose $\mathbf{w}$ so that $\mathbf{w}|_{D_0} = 0$. This fixes $\mathbf{w}_\perp$ everywhere, but leaves $w_{||}$ free elsewhere. We choose $w_{||}|_{D_1}$ such that $w_n|_{D_1} = 0$. Then

$$\frac{\partial\mathcal{A}}{\partial t}(x_0) = -\Xi(x_0) + \mathbf{w}_h\cdot\mathbf{A}_h^{\mathrm{ref}}(F(x_0)).$$

- **First term**: change in toroidal flux around the field-line due to reconnection of other field-lines.
- **Second term**: amount of poloidal flux reconnected with this field-line.

  To see this, consider the flux enclosed by a moving field-line in infinitesimal time $dt$. Ignoring change in $\mathcal{A}$ of the field-line, Stokes' Theorem gives

  $$d\Phi = -d\mathbf{x}\cdot\mathbf{A}_h^{\mathrm{ref}} \quad\Longrightarrow\quad \frac{d\Phi}{dt} = -\mathbf{w}_h\cdot\mathbf{A}_h^{\mathrm{ref}}.$$

  This is entirely determined by the endpoint displacement $d\mathbf{x}$ (due to its reconnection) and reference field on $D_1$.
- For field-lines with $\mathbf{w}_h = 0$, $\partial\mathcal{A}/\partial t$ gives exactly the GMR rate. These field-lines are the same for any choice of $\mathbf{w}$ and are precisely the **topological field-lines** of GMR theory.

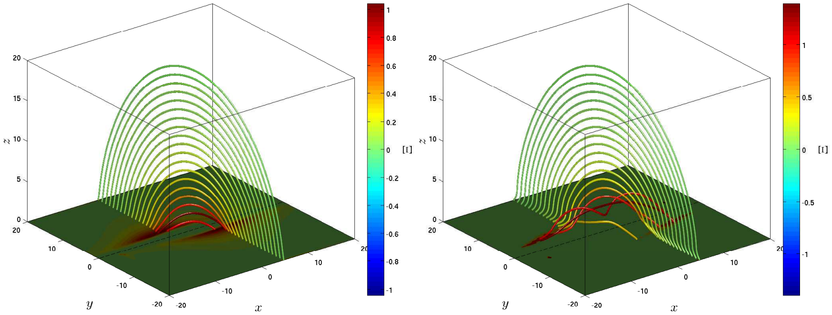
# 5. Example: Flux Rope Formation

**We illustrate with a kinematic model for the formation of a twisted flux rope by reconnection (introduced by Hesse et al., 2005 & Titov et al. 2009).**

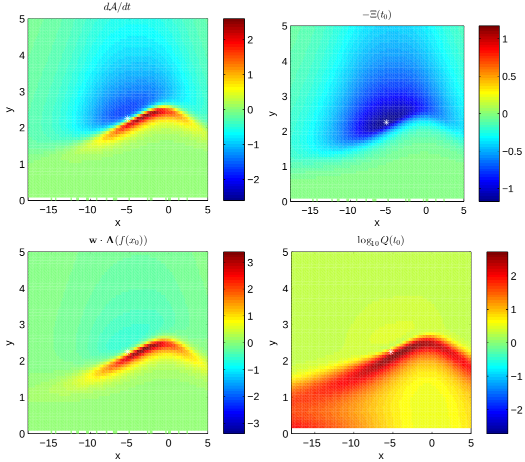
Magnetic field: $$\mathbf{B} = 3\mathbf{e}_x - \left( z + \frac{(1-z^2)t}{(1+z^2)^2(1+x^2/36)} \right) \mathbf{e}_y + y\mathbf{e}_z,$$

Electric field (to satisfy Faraday's Law): $$\mathbf{E} = \frac{z}{(1+z^2)(1+x^2/36)} \mathbf{e}_x,$$

Vector potential: $$\mathbf{A} = -\left( \frac{y^2+z^2}{2} + tE_x \right) \mathbf{e}_x + 3y\mathbf{e}_z \quad [\mathbf{A}^{\text{ref}} = \mathbf{A}(t=0)].$$



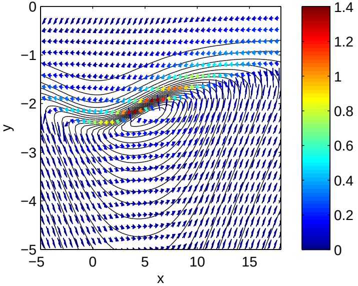

Above: Field-lines at $t = 5$ and $t = 40$, coloured by $\Xi$.




Above: Decomposition of $\partial\mathcal{A}/\partial t$ in the plane $D_0 = \{(x, y) : y > 0\}$ at $t = 5$. Bottom right shows squashing factor $Q$ (Titov *et al.*, 2002).

In this example, the $\mathbf{w}\cdot\mathbf{A}$ term dominates for large times, because $\mathbf{w}\cdot\mathbf{A} \sim -t(d\Xi/dt)$.

1. Both $-\Xi$ and $\mathbf{w}\cdot\mathbf{A}$ contribute to $\partial\mathcal{A}/\partial t$.
2. The term $\mathbf{w}\cdot\mathbf{A}$ vanishes at the location of maximum $\Xi$ (white *), so $\partial\mathcal{A}/\partial t = -\Xi$ there.
3. The "fastest reconnecting" field-lines, with largest $\mathbf{w}\cdot\mathbf{A}$, lie in the QSL (region of highest $Q$), as expected.




Above: Verification that $\mathbf{w}_\perp|_{D_1}$ (arrows) moves along contours of $\Xi$ (lines).