New Topological Constraints on Magnetic Relaxation

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with

Gunnar Hornig (Dundee)

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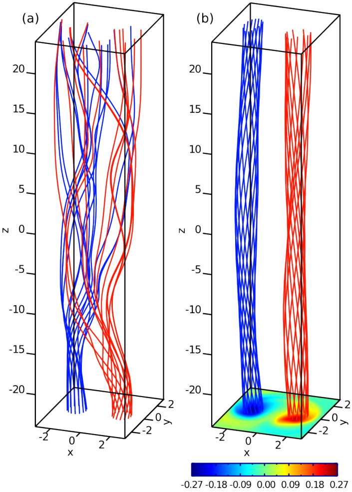
Durham University

Motivation



- Dundee simulation: relaxation of a braided magnetic field.

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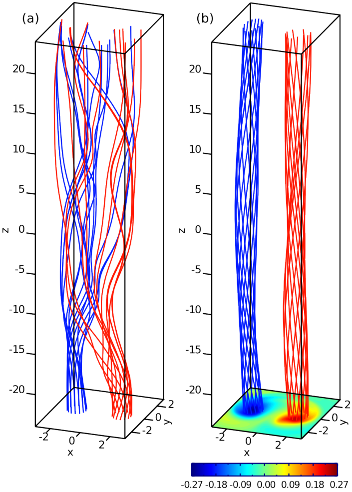

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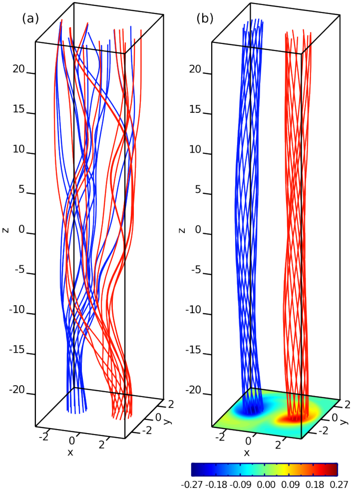

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  $\Longrightarrow$ predict final state by minimising energy subject to constrained helicity.

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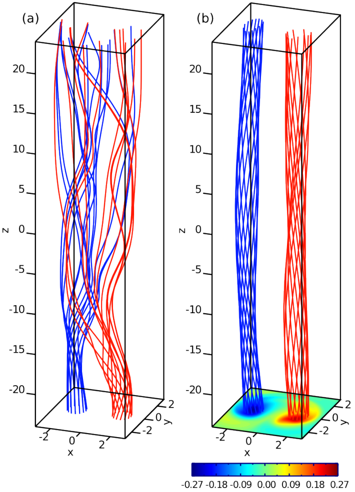

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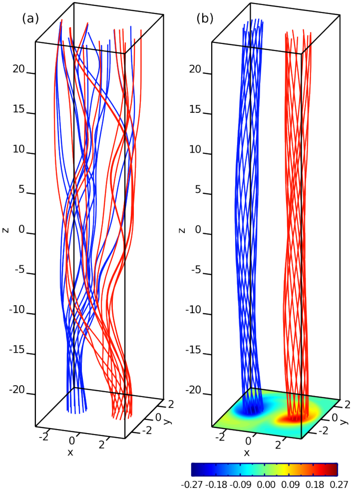

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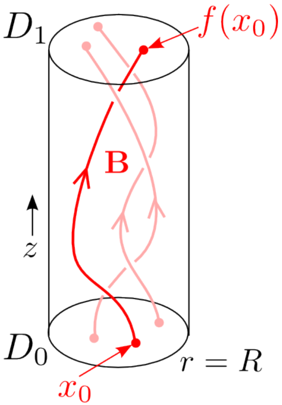
Why is the Taylor state not reached?

Previous ideas:

- Constrain multiple partial helicities over sub-volumes [Bhattacharjee & Dewar 1982; Dixon et al. 1989; Turnbull 2012]
- High-order linking of field lines [Ruzmaikin & Akhmetiev 1994; Hornig & Mayer 2002]

Magnetic Braids



- Magnetic field $B(r, \phi, z, t)$ on a cylinder, satisfying

$$B \neq 0, \qquad (1)$$

$$B_r|_{r=R} = 0, \qquad (2)$$

$$B_z|_{D_0} = B_z|_{D_1} > 0. \qquad (3)$$

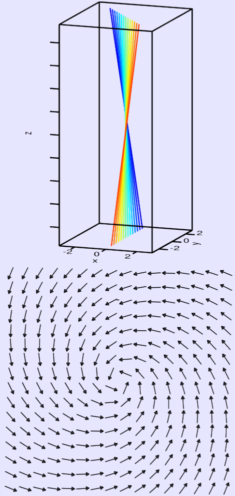
- Field line mapping

$$f : D_0 \to D_1.$$

Fixed Points

- Visualise with a "vector field" $v = f - \mathrm{id}$.

Example: uniform twist field



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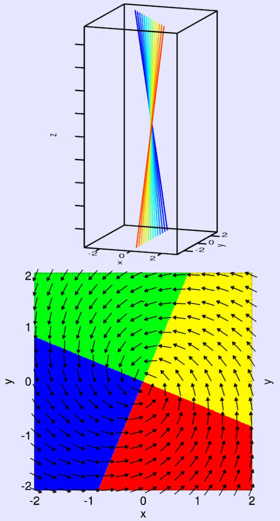
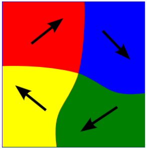


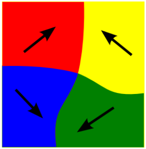
Fixed Points

- Visualise with a “vector field” $v = f - \mathrm{id}$.
- Colour map [Polymilis et al. 2003].
- Each fixed point x_0 has a Poincaré index:



$\mathrm{ind}_{x_0} f = +1$

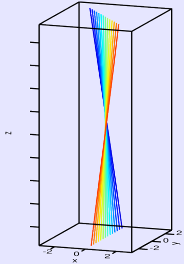
elliptic

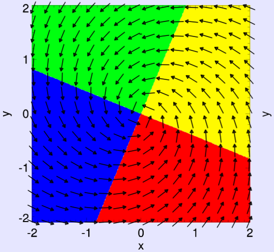


$\mathrm{ind}_{x_0} f = -1$

hyperbolic

Example: uniform twist field





Boundary Fixed Points

- To define index of fixed points on $f|_{\partial D_0}$, extend f outside D_0.
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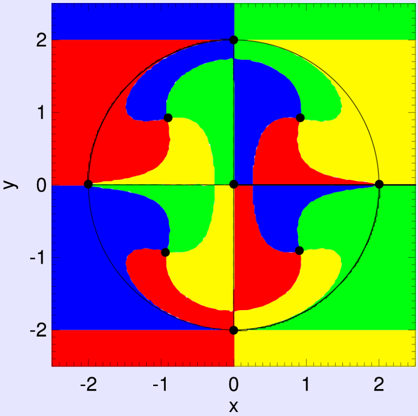

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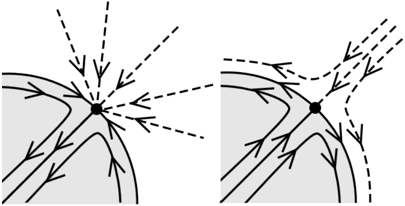
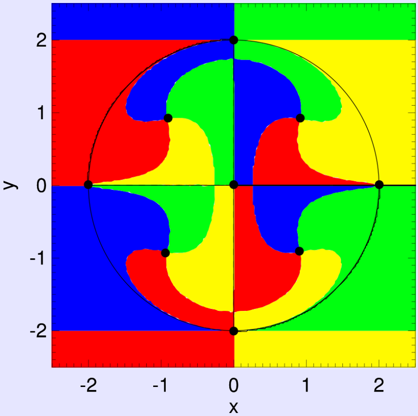
- Two types of boundary fixed point:



$\text{ind}_{\mathbf{x}_0} f = 0$ $\qquad$ $\text{ind}_{\mathbf{x}_0} f = -1$

Example:



Global Constraint

Define

- Total (net) interior index T_{int} (topological degree/Lefschetz number).
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For the disc, $T_{int} = 1 - T_{\partial}$.

Heinz Hopf

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- Total (net) boundary index T_{∂}.

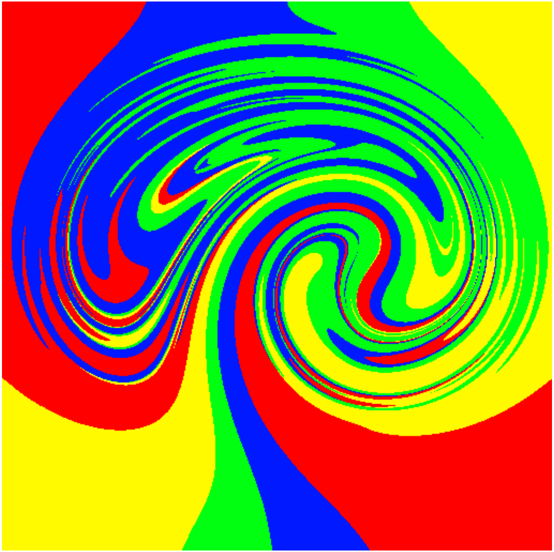
Theorem [Hopf, 1929]

For the disc, $T_{int} = 1 - T_{\partial}$.

Heinz Hopf

$\Longrightarrow$ If f remains fixed on the boundary, then T_{int} must be conserved.

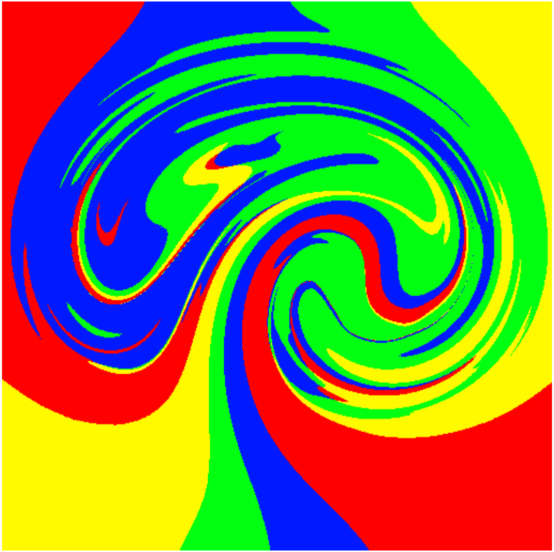
Dundee Simulation



$t = 0$

- Boundary shows that $T_{int} = 2$.

Dundee Simulation



$t = 35$

- Boundary shows that $T_{int} = 2$.

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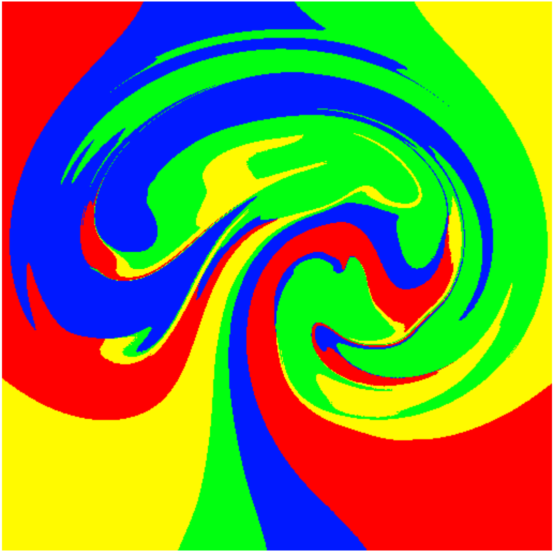

$t = 50$

- Boundary shows that $T_{int} = 2$.

Dundee Simulation

$t = 80$

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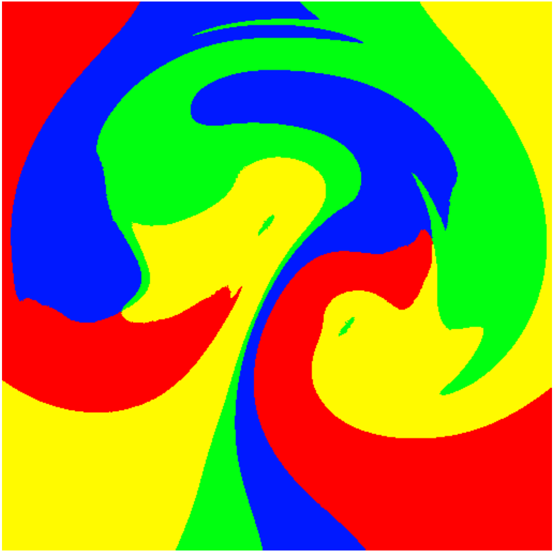
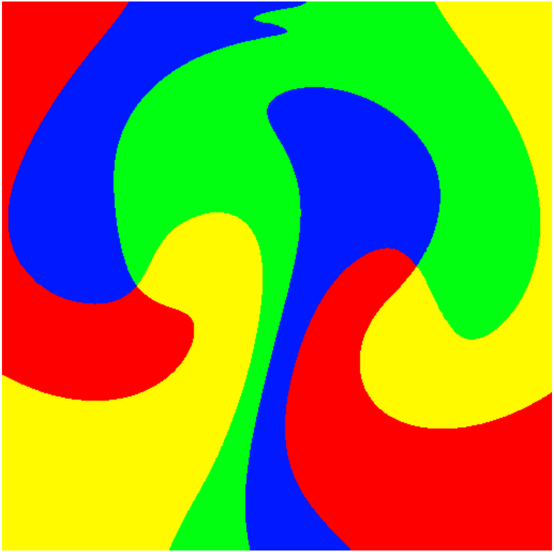


$t = 110$

- Boundary shows that $T_{int} = 2$.

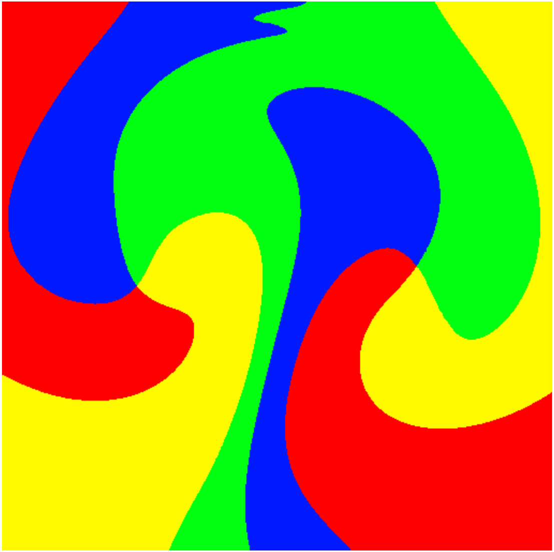
Dundee Simulation



t = 290

- Boundary shows that $T_{int} = 2$.

Dundee Simulation

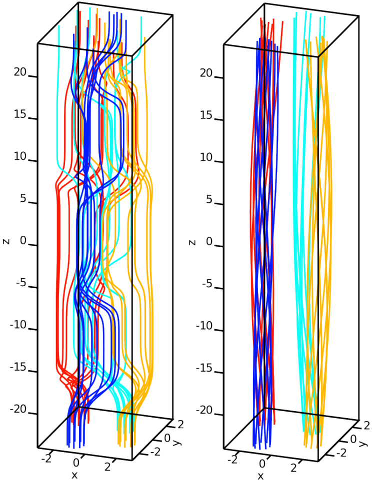


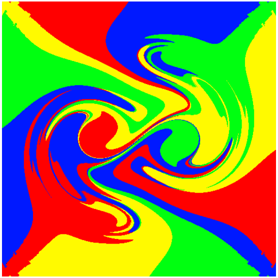
$t = 290$

- Boundary shows that $T_{\text{int}} = 2$.
- $T_{\text{int}} = 2$ is conserved, explaining failure to reach Taylor state.

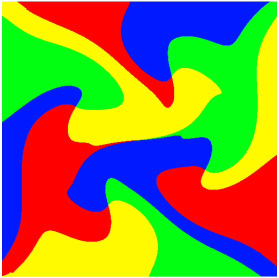
The Silver Braid

- New initial state with $T_{int} = 3$ (inspired by "silver mixer" [Finn & Thiffeault 2011])





Initial



Final

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Denote the total index of $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{\text{n times}}$ by T^n_{int}.

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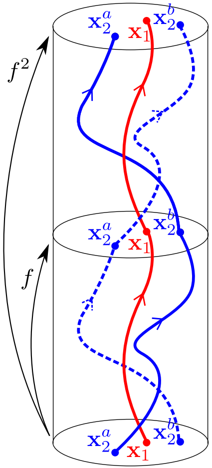

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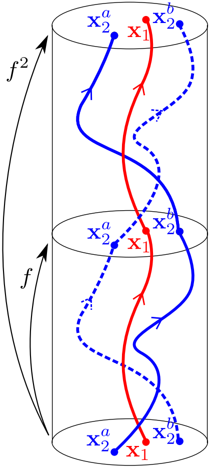

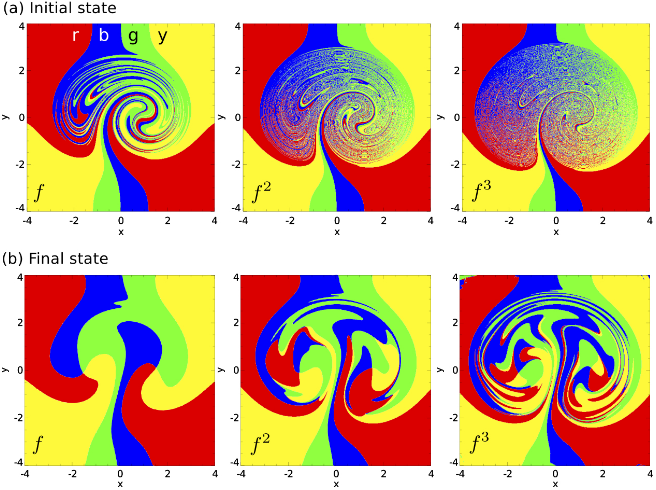

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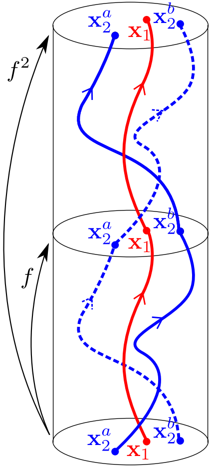

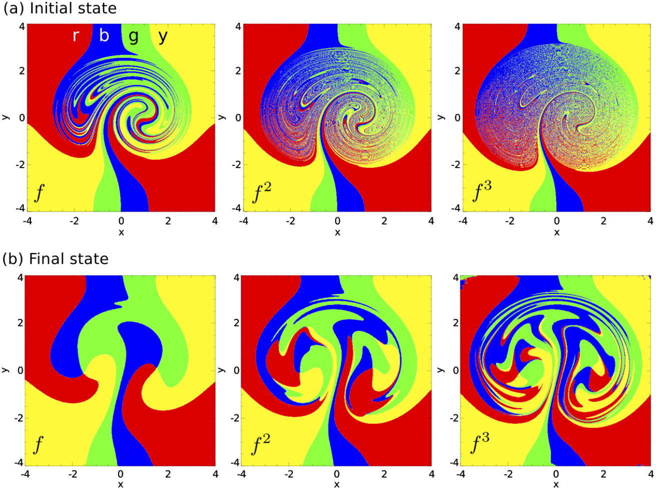

Do these additional invariants constrain our predicted relaxed state?

Non-independence of Constraints

Usually "no"...

Theorem (Yeates & Hornig, 2011)

Suppose f has a finite number of fixed points at all iterations, and $\mathrm{ind}_{x_0} f^q \in \{-1, 0\}$ for every periodic point x_0 on ∂D_0. Then $T^q_{\mathrm{int}} = T^1_{\mathrm{int}}$ for all $q \in \mathbb{N}$ unless $T^1_{\mathrm{int}} = 1$ and $f|_{\partial D_0}$ has a rational rotation number greater than 1.

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Sketch proof:

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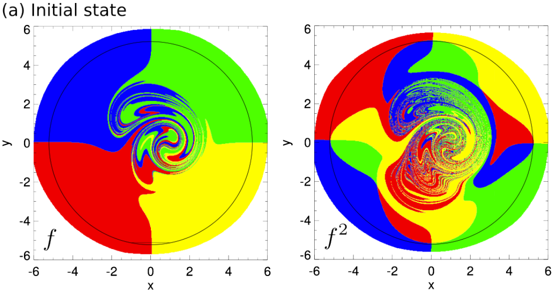
$$T^1_{\mathrm{int}} > 1 \implies T^1_{\partial} \neq 0 \quad \text{(all boundary fixed points already present at } f^1\text{)}$$

$$T^1_{\mathrm{int}} = 1 \implies T^1_{\partial} = 0 \implies \text{minimal period could be higher (if rotation number} \in \mathbb{Q}\text{).}$$

Exceptional Case

- Initial field with

$$T_{\rm int}^{1,3,5,\ldots} = 1,$$
$$T_{\rm int}^{2,4,6,\ldots} = 3.$$



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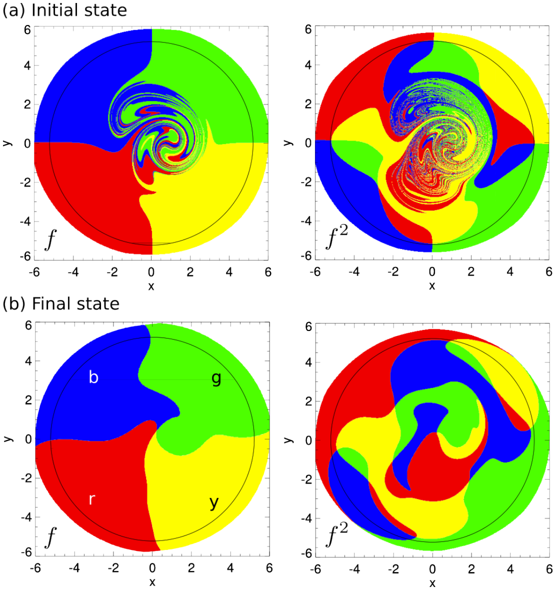

Conclusion

- Topological degree of the field line mapping can constrain the turbulent relaxation of a magnetised plasma.
- Applies to any continuous evolution of B providing that the field remains ideal on the boundary.
- If the degree of the initial state differs from that of the Taylor state, then the Taylor state will not be reached in the dynamical relaxation.
- There can be up to one further constraint from higher iterations of the field line mapping, but only for certain initial states with degree 1.

References

1. Yeates, Hornig & Wilmot-Smith, PRL 105, 085002 (2010).
2. Yeates & Hornig, J. Phys. A 44, 265501 (2011).
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*Use periodic field lines to define a global reconnection rate.