A COUNTEREXAMPLE TO BATSON'S CONJECTURE.

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Abstract. We show that the torus knot $T_{4,9}$ bounds a smooth Möbius band in the 4-ball, giving a counterexample to Batson's non-orientable analogue of Milnor's conjecture on the smooth slice genera of torus knots.

Batson's conjecture says that the smooth non-orientable 4-ball genus of a torus knot is realized by a simple construction. This is analogous to Milnor's conjecture (verified by Kronheimer-Mrowka [3]) that the smooth orientable 4-ball genus of a torus knot is realized by the surface obtained from applying Seifert's algorithm to a standard diagram of the knot.

The conjecture.

Let $T_{p,q} \subset S^3$ be the (p,q) torus knot for $p > q \geq 2$, and let $D_{p,q}$ be the usual q-stranded braid closure diagram of $T_{p,q}$. Adding a blackboard-framed 1-handle (the interior of whose core is disjoint from $D_{p,q}$) between the first two strands of $D_{p,q}$ results in a simpler torus knot, whose usual braid closure diagram we then consider. Repeating this procedure eventually arrives at the unknot, which may be capped off in the 4-ball to give a surface $F_{p,q} \subset B^4$ with $\partial F_{p,q} = T_{p,q}$. Batson conjectured [1] that $b_1(F_{p,q})$ is minimal among the first Betti numbers of non-orientable smooth surfaces in the 4-ball with boundary $T_{p,q}$.

We heard of this conjecture in a talk by Van Cott who, together with Jabuka, has verified it in many cases [2].

A counterexample.

Figure 1. On the left is shown the torus knot $T_{4,9}$. In the middle we have added two 1-handles resulting in the unknot - this describes the surface $F_{4,9} \subset B^4$ which has $b_1(F_{4,9}) = 2$. On the right we show how one may add a single 1-handle to $T_{4,9}$ to result in the knot 6_1, which is smoothly slice, thus giving a surface $\Sigma \subset B^4$ with $\partial\Sigma = T_{4,9}$ and $b_1(\Sigma) = 1$.

References

1. J. Batson, *Nonorientable slice genus can be arbitrarily large*, Math. Res. Lett. **21** (2014), no. 3, 423–436.
2. C. A. Van Cott and S. Jabuka, *On a nonorientable analogue of the Milnor conjecture*, ArXiv e-print 1809.017793 (2019).
3. P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. II*, Topology **34** (1995), no. 1, 37–97.

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